

KERNEL CONVERGENCE ESTIMATES FOR DIFFUSIONS WITH CONTINUOUS COEFFICIENTS

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ABSTRACT. We are interested in the kernel of one-dimensional diffusion equations with continuous coefficients as evaluated by means of explicit discretization schemes of uniform step $h > 0$ in the limit as $h \rightarrow 0$. We consider both semidiscrete triangulations with continuous time and explicit Euler schemes with time step so small that the Courant condition is satisfied. We find sharp uniform bounds for the convergence rate as a function of the degree of smoothness which we conjecture. The bounds also apply to the time derivative of the kernel and its first two space derivatives. Our proof is constructive and is based on a new technique of path conditioning for Markov chains and a renormalization group argument. Convergence rates depend on the degree of smoothness and Hölder differentiability of the coefficients. We find that the fastest convergence rate is of order $O(h^2)$ and is achieved if the coefficients have a bounded second derivative. Otherwise, explicit schemes still converge for any degree of Hölder differentiability except that the convergence rate is slower. Hölder continuity itself is not strictly necessary and can be relaxed by an hypothesis of uniform continuity.

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1. INTRODUCTION AND NOTATIONS

Consider a pair of backward and forward one-dimensional diffusion equations of the form

$$(1.1) \quad \frac{\partial}{\partial t} f(x; t) + \mathcal{L}_x^0 f(x; t) = 0, \quad \frac{\partial}{\partial t} g(y; t) = \mathcal{L}_y^{0*} g(y; t)$$

where

$$(1.2) \quad \mathcal{L}_x^0 = \frac{1}{2} \sigma(x)^2 \frac{\partial^2}{\partial x^2} + \mu(x) \frac{\partial}{\partial x}.$$

and its adjoint formally acts as follows:

$$(1.3) \quad (\mathcal{L}_y^{0*} \phi)(y) = \frac{1}{2} \frac{\partial^2}{\partial y^2} (\sigma(y)^2 \phi(y)) - \frac{\partial}{\partial y} (\mu(y) \phi(y)).$$

on a test function ϕ . These equations are defined on the bounded interval $A = [-L, L] \subset \mathbb{R}$ where $0 < L < \infty$. For simplicity, we assume periodic boundary conditions and identify the two boundary points $\pm L$ with each other.

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In most of the paper, the coefficients $\sigma(x)$ and $\mu(x)$ are assumed to be Hölder differentiable. More precisely, if $\alpha \in (0, 1]$, $k \in \mathbb{N}$ where $\mathbb{N} = \{0, 1, \dots\}$ and the function $\phi(x) \in \mathcal{C}^k(A)$ has $k \geq 1$ continuous derivatives, then one says that ϕ is Hölder differentiable of order (k, α) if there exists a constant $c > 0$ such that

$$(1.4) \quad |\phi^{(k)}(x) - \phi^{(k)}(y)| \leq cd(x, y)^\alpha$$

uniformly for all $x, y \in A$. In case $k = 0$ the function is called Hölder continuous. The distance is defined consistently with the assumed periodic boundary conditions and is given by

$$(1.5) \quad d(x, y) = \min_n |x - y - 2Ln|.$$

The linear space of Hölder continuous or Hölder differentiable periodic functions of order (k, α) on A is denoted with $\mathcal{C}^{k, \alpha}(A)$. We are interested in the case where $\sigma^2 \in \mathcal{C}^{k, \alpha}(A)$ and $\mu \in \mathcal{C}^{j, \beta}(A)$ with $k + \alpha > 0$ and $j + \beta > 0$.

The hypothesis of Hölder continuity can be relaxed slightly by assuming uniform continuity instead, i.e. that both $\sigma(x)^2$ and $\mu(x)$ satisfy a bound of the form

$$(1.6) \quad |\phi(x) - \phi(y)| \leq c\rho(d(x, y))$$

where $\rho(d)$ is a non-decreasing function such that $\lim_{d \downarrow 0} \rho(d) = 0$.

Let $u_0(x, y; t)$ denote a kernel of equation (1.17), i.e. a weak solution of the forward equation

$$(1.7) \quad \frac{\partial}{\partial t} u_0(x, y; t) = \mathcal{L}_y^{0*} u_0(x, y; t)$$

where the operator \mathcal{L}_y^{0*} acts on the y coordinate and the following initial time condition is satisfied:

$$(1.8) \quad \lim_{t \downarrow 0} u_0(x, y; t) = \delta(x - y).$$

The kernel $u_0(x, y; t)$ formally satisfies also the backward equation

$$(1.9) \quad \frac{\partial}{\partial t} u_0(x, y; t) + \mathcal{L}_x^0 u_0(x, y; t) = 0$$

where the operator \mathcal{L}_x^0 acts on the x coordinate.

We are interested in existence, uniqueness, smoothness and approximation schemes for the kernel $u_0(x, y; t)$, its first two space derivatives with respect to the x variable and its first time derivative $\partial_t u_0(x, y; t)$. As a byproduct of this analysis, we also find conclusions about the convergence of $\mathcal{L}_x^0 u_0(x, y; t)$ and $\mathcal{L}_y^{0*} u_0(x, y; t)$, as both expressions equal the first time derivative.

Diffusion equations are one of the single most studied topics in the literature. Existence and uniqueness questions for the kernel were address in (Kolmogorov 1931), (Feller 1936), (Hille 1948), (Yosida 1951) and (Ito 1957). A classification of all the possible boundary conditions is in (Feller 1952). The case of Hölder continuous coefficients was resolved in (Philips 1961) based on methods in (Friedrichs 1958) and (Lax and Phillips 1960). The hypothesis of Hölder continuity was relaxed to uniform continuity in (Fabes and Riviere 1966) and (Stroock and Varadhan 1969). Stroock and Varadhan also introduce a new probabilistic framework where existence is proved by reduction to the so-called martingale problem and a compactness argument, thus shifting the attention from the kernel itself to the underlying measure space.

The existence of a weak limit of continuous time Markov chains as $h_m \downarrow 0$ was established in (Sova 1967) and (Kurtz 1969) by using operator semigroup methods, see also the book (Ethier and Kurtz 1986) for a review. Convergence in the semigroup sense takes place if the limit

$$(1.10) \quad (T_t \phi)(x) = \lim_{m \rightarrow \infty, m \geq n} \sum_{y \in h_m Z \cap A} u_{h_m}(x, y; t) \phi(y)$$

exists for all test functions $\phi \in \mathcal{C}^\infty(A)$, uniformly for all $x \in A_n, n \geq 0$. A key result is that a necessary and sufficient condition for this limit to exist and define a semigroup T_t is that generators converge also in the same Banach space, i.e. that also the limit

$$(1.11) \quad \lim_{h_m \downarrow 0} \mathcal{L}_x^{h_m} \phi = \mathcal{L}_x^0 \phi$$

exists in the uniform norm for all test functions $\phi \in \mathcal{C}^\infty(A)$. See (Ethier and Kurtz 1986) for a precise statement with all the technical conditions and a proof. In (Stroock and Varadhan 1979) convergence is reconsidered again by reduction to the martingale problem.

The problem has also been studied extensively in the numerical analysis literature. Explicit and implicit Euler schemes where coefficients are smooth and the data is rough in the sense that it belongs to a L^2 space have been considered by several authors. In the case that the Markov generator is symmetric and time independent, one can make use of a spectral representation as in (Baker *et al.* 1977) and with greater effort such methods may also be used for more general situations, see (Suzuki 1978). In (Luskin and Rannacher 1978), a parabolic duality argument is used to show convergence for the standard Galerkin method. (Mingyou and Thomee 1982) use a simpler argument based on energy estimates. In (Palencia 1996) one finds convergence bounds in maximum norm assuming the initial condition is uniformly bounded and coefficients are constant.

In this article, we revisit this classic theme by considering the problem of obtaining the kernel constructively as a limit of increasingly fine triangulations schemes and in assessing the rate of convergence with pointwise bounds on the kernel itself. More precisely, let $h_m = L2^{-m}$, $m \in \mathbb{N}$ and let $A_m h_m \mathbb{Z} \cap A$. Consider the sequence of operators

$$(1.12) \quad \mathcal{L}_x^m = \frac{\sigma(x)^2}{2} \Delta_x^m + \mu(x) \nabla_x^m.$$

defined on the 2^{m+1} -dimensional space of all periodic functions $f_m : A_m \rightarrow \mathbb{R}$, where

$$(1.13) \quad \nabla_x^m f(x) = \frac{f(x+h_m) - f(x-h_m)}{2h_m}.$$

and

$$(1.14) \quad \Delta_x^m f(x) = \frac{f(x+h_m) + f(x-h_m) - 2f(x)}{h_m^2}$$

These definitions also apply to the boundary points by periodicity. We assume that $m \geq m_0$ where m_0 is the least integer such that

$$(1.15) \quad \frac{\sigma^2(x)}{2h_m^2} > \frac{|\mu(x)|}{2h_m}$$

for all $m \geq m_0$ and all $x \in A_m$.

Let $u_m(x, y; t)$ denote the kernel of equation (1.17), i.e. the solution of the (forward) equation

$$(1.16) \quad \frac{\partial}{\partial t} u_m(x, y; t) = \mathcal{L}_y^{m*} u_m(x, y; t)$$

where the operator \mathcal{L}_y^{m*} acts on the y coordinate and the following initial time condition is satisfied:

$$(1.17) \quad \lim_{t \downarrow 0} u_m(x, y; t) = h_m^{-1} \delta_m(x - y).$$

Here,

$$(1.18) \quad \delta_m(x - y) = \begin{cases} 1 & \text{if } x = y \pmod{2L} \\ 0 & \text{otherwise.} \end{cases}$$

Since (1.17) is a finite system of linear ordinary differential equations, the solution exists and is unique for all times. The kernel $u_m(x, y; t)$ satisfies also the backward equation

$$(1.19) \quad \frac{\partial}{\partial t} u_m(x, y; t) + \mathcal{L}_x^m u_m(x, y; t) = 0$$

where the operator \mathcal{L}_x^m acts on the x coordinate. Using functional calculus notations for the exponential of a matrix, we also have that

$$(1.20) \quad u_m(x, y; t) = \frac{1}{h_m} \exp(t\mathcal{L}^m)(x, y).$$

Our main result can be stated as follows:

Theorem 1. *Suppose that $\sigma^2 \in \mathcal{C}^{k,\alpha}$ and $\mu \in \mathcal{C}^{j,\beta}$ and let*

$$(1.21) \quad \gamma = \min\{2, k + \alpha, j + \beta\}.$$

Assume that $\gamma > 0$ and that $\sigma(x) \geq \Sigma_0$ for some constant $\Sigma_0 > 0$. Then there is a constant $c > 0$ such that for all $m' \geq m \geq m_0$ and all $x, y \in A_m$ the following inequalities hold:

$$(1.22) \quad \begin{aligned} (i) \quad & |u_m(x, y; t) - u_{m'}(x, y; t)| \leq ch_m^\gamma \\ (ii) \quad & |\partial_t u_m(x, y; t) - \partial_t u_{m'}(x, y; t)| \\ &= |\mathcal{L}_y^{m*} u_m(x, y; t) - \mathcal{L}_y^{m'*} u_{m'}(x, y; t)| \\ &= |\mathcal{L}_x^m u_m(x, y; t) - \mathcal{L}_x^{m'} u_{m'}(x, y; t)| \leq ch_m^\gamma. \end{aligned}$$

(1.23)

A version of this theorem under slightly weaker conditions can be formulated as follows:

Theorem 2. *Suppose that $\sigma(x)^2$ and $\mu(x)$ are uniformly continuous functions in A . Let the function $\rho(d)$ be non-decreasing and be such that $\lim_{d \downarrow 0} \rho(d) = 0$ and equation (1.6) holds. Assume also that $\sigma(x) \geq \Sigma_0$ for some constant $\Sigma_0 > 0$. Then there is a constant $c > 0$ such that for all $m' \geq m \geq m_0$ and all $x, y \in A_m$ the following inequalities hold:*

$$(1.24) \quad \begin{aligned} (i) \quad & |u_m(x, y; t) - u_{m'}(x, y; t)| \leq c\rho(h_m) \\ (ii) \quad & |\partial_t u_m(x, y; t) - \partial_t u_{m'}(x, y; t)| \\ &= |\mathcal{L}_y^{m*} u_m(x, y; t) - \mathcal{L}_y^{m'*} u_{m'}(x, y; t)| \\ &= |\mathcal{L}_x^m u_m(x, y; t) - \mathcal{L}_x^{m'} u_{m'}(x, y; t)| \leq c\rho(h_m). \end{aligned}$$

(1.25)

Next, we consider the case where also time is discretized and prove the following result:

Theorem 3. *Suppose that σ^2 and μ satisfy equations of the form (1.6) with a non-decreasing function $\rho(d)$ such that $\lim_{d \downarrow 0} \rho(d) = 0$. Assume also that $\sigma(x) \geq \Sigma_0$ for some constant $\Sigma_0 > 0$. Consider the discretized kernel*

$$(1.26) \quad u_m^{\delta t}(x, y; t) = h_m^{-1} (1 + \delta t \mathcal{L}^m)^{\lceil \frac{t}{\delta t} \rceil} (x, y; t).$$

where \mathcal{L}^m is the operator in (1.12) and δt_m is so small that

$$(1.27) \quad \min_{x \in A_m} 1 + \delta t_m \mathcal{L}^m(x, x) > 0$$

Assume that boundary conditions are periodic and that the ratio $\frac{t}{\delta t} = N$ is an integer. Then here is a constant $c > 0$ such that the following bounds hold for all $m \geq m_0$ and all $x, y \in A_m$:

$$(1.28) \quad \begin{aligned} (i) \quad & |u_m(x, y; t) - u_m^{\delta t}(x, y; t)| \leq ch_m^2 \\ (ii) \quad & \left| \partial_t u_m(x, y; t) - \frac{u_m^{\delta t}(x, y; t + \delta t) - u_m^{\delta t}(x, y; t)}{\delta t} \right| \\ &= |\mathcal{L}_y^{m*} u_m(x, y; t) - \mathcal{L}_y^{m*} u_m^{\delta t}(x, y; t)| \\ &= |\mathcal{L}_x^{m*} u_m(x, y; t) - \mathcal{L}_x^{m*} u_m^{\delta t}(x, y; t)| \leq ch^2 \end{aligned}$$

(1.29)

The paper is organized as follows. In Section 2 we consider the case of Brownian motion and review a result in (Albanese and Mijatovic 2006) which establishes the theorems above in this simple particular case where Fourier analysis in the space direction can be used to carry out a precise calculation. In Section 3, we consider the case of a diffusion where both the volatility and the drift have two bounded derivatives. In this case, we make use of time-homogeneity and carry out a Fourier transform in the time direction after path conditioning. In Section 4, we extend the derivation to the case of non-smooth coefficients. Section 5 is dedicated to the case where time is discretized and we prove Theorem 3.

2. CONSTANT COEFFICIENTS

In this Section, we prove Theorem 1 in the special case where the volatility and the drift coefficients are constant, i.e.

$$(2.1) \quad \mathcal{L}^m_x = \mu \nabla_x^m + \frac{1}{2} \sigma^2 \Delta_x^m.$$

It suffices to consider the case $m' = m + 1$. Let B_m be the Brillouin zone defined as follows:

$$(2.2) \quad B_m = \left\{ -\frac{2^{m-1}\pi}{L} + \frac{k\pi}{L}, k = 0, \dots, 2^m - 1 \right\}$$

Let $\mathcal{F}_m : \ell^2(A_m) \rightarrow \ell^2(B_m)$ be the Fourier transform operator defined so that:

$$(2.3) \quad \hat{f}(p) \equiv \mathcal{F}_m(f)(p) = h_m \sum_{x \in A_m} f(x) e^{-ipx}$$

for all $p \in B_m$. The inverse Fourier transform is given by

$$(2.4) \quad \mathcal{F}_m^{-1}(\hat{f})(x) = \frac{1}{2L} \sum_{p \in B_m} \hat{f}(p) e^{ipx}.$$

The Fourier transformed generator is diagonal and is given by the operator of multiplication by

$$(2.5) \quad \hat{\ell}^m(p) = \mathcal{F}_m \mathcal{L}^m \mathcal{F}_m^{-1}(p, p) = -i\mu \frac{\sin h_m p}{h_m} + \sigma^2 \frac{\cos h_m p - 1}{h_m^2}.$$

We have

$$(2.6) \quad u_m(x, y; t) = \frac{1}{2L} \sum_{p \in B_m} e^{t\hat{\ell}^m(p)} e^{ip(y-x)}.$$

Using this Fourier series representation, we find

$$(2.7) \quad \begin{aligned} & |u_m(x, y; t) - u_{m+1}(x, y; t)| \\ & \leq \frac{1}{2L} \left| \sum_{p \in B_m} \left(e^{t\hat{\ell}^m(p)} - e^{t\hat{\ell}^{m+1}(p)} \right) e^{ip(y-x)} \right| + \frac{1}{2L} \left| \sum_{p \in B_{m+1} \setminus B_m} e^{t\hat{\ell}^{m+1}(p)} e^{ip(y-x)} \right|. \end{aligned}$$

Let

$$(2.8) \quad K_m = \sqrt{\frac{|\log h_{m+1}|}{\sigma^2 t}}.$$

If h_m is small enough, i.e. if m_0 is sufficiently large, we have that

$$(2.9) \quad \frac{1}{2L} \left| \sum_{p \in B_m, |p| \geq K_m} e^{t\hat{\ell}^m(p)} e^{ip(y-x)} \right| \leq \frac{1}{2L} \sum_{p \in B_m, |p| \geq K_m} e^{t\Re(\hat{\ell}^m(p))} \leq c \exp\left(t\sigma^2 \frac{\cos h_m K_m - 1}{h_m^2} \right) \leq ch_m^2.$$

where $\Re(a)$ denotes the real part of $a \in \mathbb{C}$ and c denotes a generic constant. Similarly

$$\begin{aligned} \frac{1}{2L} \left| \sum_{p \in B_{m+1}, |p| \geq K_m} e^{t\hat{\ell}^{m+1}(p)} e^{ip(y-x)} \right| &\leq \frac{1}{2L} \sum_{p \in B_m, |p| \geq K_m} e^{t\Re(\hat{\ell}^{m+1}(p))} \\ &\leq c \exp\left(t\sigma^2 \frac{\cos h_{m+1}K - 1}{h_{m+1}^2}\right) \leq ch_{m+1}^2 \end{aligned}$$

(2.10)

Since

$$(2.11) \quad \frac{1}{2}h^2p^3 - \frac{1}{8}h^4p^5 \leq \frac{\sin hp}{h} - \frac{\sin 2hp}{2h} \leq \frac{1}{2}h^2p^3$$

and

$$(2.12) \quad -\frac{1}{8}h^2p^4 \leq \frac{\cos hp - 1}{h^2} - \frac{\cos 2hp - 1}{(2h)^2} \leq -\frac{1}{8}h^2p^4 + \frac{1}{48}h^4p^6.$$

we find that if $|p| \leq \frac{\sqrt{2}}{h}$ then

$$(2.13) \quad |\hat{\ell}^m(p) - \hat{\ell}^{m+1}(p)| \leq \frac{\mu}{4}h^2|p|^3 + \frac{\sigma^2}{16}h^2p^4.$$

Moreover, since

$$(2.14) \quad -\frac{1}{2}p^2 \leq \frac{\cos hp - 1}{h} \leq -\frac{1}{2}p^2 + \frac{1}{24}h^2p^4$$

we conclude that in case $|p| \leq h^{-1}\sqrt{\frac{2}{3}}$, the following inequality holds:

$$(2.15) \quad \frac{\cos hp - 1}{h} \leq -\frac{1}{4}p^2$$

Hence, if m_0 is large enough, we find

$$\begin{aligned} \frac{1}{2L} \left| \sum_{p \in B_m, |p| \leq K} \left(e^{t\hat{\ell}^m(p)} - e^{t\hat{\ell}^{m+1}(p)} \right) e^{ip(y-x)} \right| \\ \leq \frac{1}{2L} \sum_{p \in B_m, |p| \leq K} e^{-\frac{1}{4}p^2} \left(e^{\frac{\mu t}{4}h_m^2|p|^3 + \frac{\sigma^2 t}{16}h_m^2p^4} - 1 \right) \\ \leq \frac{1}{2L} \sum_{p \in B_m, |p| \leq K} e^{-\frac{1}{4}p^2} \left(\frac{\mu t}{4}h_m^2|p|^3 + \frac{\sigma^2 t}{16}h_m^2p^4 \right) \leq ch_m^2 \end{aligned}$$

for some constant $c > 0$ independent of m . This concludes the proof of convergence for the kernel in the special case of constant coefficients.

To estimate the first derivative, notice that

$$(2.17) \quad \nabla u_m(x, y; t) = \frac{1}{L} \sum_{p \in B_m} e^{t\hat{\ell}^m(p)} \frac{\sin ph_m}{h_m} e^{ip(y-x)},$$

and

$$\begin{aligned} &|u_m(x, y; t) - u_{m+1}(x, y; t)| \\ &\leq \frac{1}{2L} \left| \sum_{p \in B_m} \left(e^{t\hat{\ell}^m(p)} \frac{\sin ph_m}{h_m} - e^{t\hat{\ell}^{m+1}(p)} \frac{\sin ph_{m+1}}{h_{m+1}} \right) e^{ip(y-x)} \right| \\ &+ \frac{1}{2L} \left| \sum_{p \in B_{m+1} \setminus B_m} e^{t\hat{\ell}^{m+1}(p)} e^{ip(y-x)} \right|. \end{aligned}$$

(2.18)

Let

$$(2.19) \quad K_m = 2\sqrt{\frac{|\log h_{m+1}|}{\sigma^2 t}}.$$

If h_m is small enough, we have that

$$(2.20) \quad \begin{aligned} \frac{1}{2L} \left| \sum_{p \in B_m, |p| \geq K_m} e^{t\hat{\ell}^m(p)} \frac{\sin ph_m}{h_m} e^{ip(y-x)} \right| &\leq \frac{1}{2L} \sum_{p \in B_m, |p| \geq K_m} e^{t\Re(\hat{\ell}^m(p))} \frac{\sin ph_m}{h_m} \\ &\leq c \left| \frac{\sin Kh_m}{h_m} \right| \exp\left(t\sigma^2 \frac{\cos Kh_m - 1}{h_m^2}\right) \leq ch_m^2. \end{aligned}$$

where c denotes a generic constant. Similarly

$$(2.21) \quad \frac{1}{2L} \left| \sum_{p \in B_{m+1}, |p| \geq K_m} e^{t\hat{\ell}^{m+1}(p)} e^{ip(y-x)} \right| \leq ch^2.$$

If m is large enough, we also find

$$(2.22) \quad \begin{aligned} \frac{1}{2L} \left| \sum_{p \in B_m, |p| \leq K_m} \left(\frac{\sin ph_m}{h_m} e^{t\hat{\ell}^m(p)} - \frac{\sin ph_{m+1}}{h_{m+1}} e^{t\hat{\ell}^{m+1}(p)} \right) e^{ip(y-x)} \right| \\ \leq \frac{1}{2L} \sum_{p \in B_m, |p| \leq K_m} \left| \frac{\sin ph_m}{h_m} \right| e^{-\frac{1}{4}p^2} \left(e^{\frac{\mu t}{4} h_m^2 |p|^3 + \frac{\sigma^2 t}{16} h_m^2 p^4} - 1 \right) \\ + e^{-\frac{1}{4}p^2} \left| \frac{\sin ph_{m+1}}{h_{m+1}} - \frac{\sin ph_m}{h_m} \right| \leq ch_m^2 \end{aligned}$$

for some constant $c > 0$ independent of m . This concludes the proof of the bound of the first derivative. The second derivative can be derived in a similar way.

Finally, consider the following Fourier representation for the discretized kernel

$$(2.23) \quad u_m^{\delta t}(x, y; t) = \frac{1}{L} \sum_{p \in B_m} \left(1 + \delta t \hat{\ell}^m(p) \right)^{\frac{t}{\delta t}} e^{ip(y-x)}.$$

Consider the formula

$$(2.24) \quad \left(1 + \delta t \hat{\ell}^m(p) \right)^{\frac{t}{\delta t}} = \exp\left(t \log(1 + \hat{\ell}^m(p))\right).$$

and let's represent the difference between the discrete and continuous time kernels as follows:

$$(2.25) \quad \begin{aligned} &|u_m(x, y; t) - u_m^{\delta t}(x, y; t)| \\ &\leq \frac{1}{2L} \left| \sum_{p \in B_m} \left(\exp(t\hat{\ell}^m(p)) - \exp\left(\frac{t}{\delta t} \log(1 + \delta t \hat{\ell}^m(p))\right) \right) e^{ip(y-x)} \right| \\ &\leq \frac{1}{2L} \sum_{p \in B_m, |p| \leq K_m} e^{-\frac{1}{4}p^2} \left| \exp\left(\frac{t}{\delta t} \log(1 + \delta t \hat{\ell}^m(p))\right) - t\hat{\ell}^m(p) \right| \\ &\quad + \frac{1}{2L} \sum_{p \in B_m, |p| \geq K_m} \left| \exp(t\hat{\ell}^m(p)) \right| + \frac{1}{2L} \sum_{p \in B_m, |p| \geq K_m} \left| \exp\left(\frac{t}{\delta t} \log(1 + \delta t \hat{\ell}^m(p))\right) \right| \end{aligned}$$

where K_m is chosen as in (2.8). The very same bounds above lead to the conclusion that this difference is $\leq ch_m^2$.

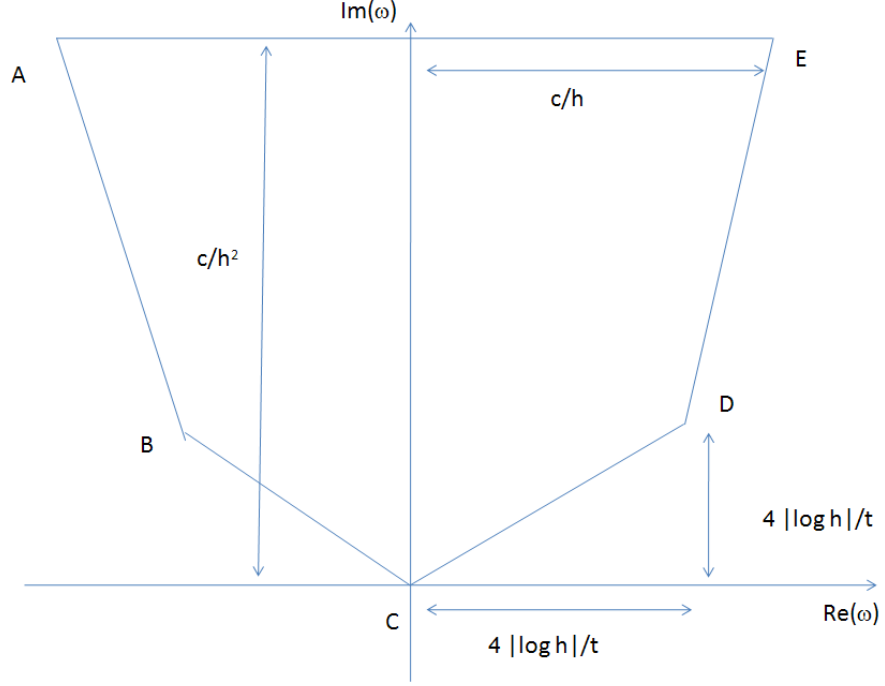


FIGURE 1. Contour of integration for the integral in (3.52). \mathcal{C}_+ is the countour joining the point D to the points E, A, B . \mathcal{C}_- is the countour joining the point B to C to D .

3. SMOOTH COEFFICIENTS

In this section, we prove Theorem 1 in the case where the drift and volatility are both of class $\mathcal{C}^{(3,0)}$, i.e. they depend smoothly on the space coordinate but not on the time coordinate.

Let us introduce the following two constants characterizing the volatility function:

$$(3.1) \quad \Sigma_0 = \inf_{x \in A_m} \sigma(x), \quad \Sigma_1 = \sup_{x \in A_m} \sqrt{\sigma(x)^2 + h_m |\mu(x)|}.$$

and let

$$(3.2) \quad M = \sup_{x \in A_m} |\mu(x)|.$$

Due to our assumptions, we have that $\Sigma_0 > 0$ and $\Sigma_1, M < \infty$.

A *symbolic path* $\gamma = \{\gamma_0, \gamma_1, \gamma_2, \dots\}$ is an infinite sequence of sites in A_m such that $\gamma_j \neq \gamma_{j-1}$ for all $j = 1, \dots$. Let Γ_m be the set of all symbolic paths in A_m . The kernel of the diffusion process admits the following representation in terms of a summation over symbolic paths

$$(3.3) \quad u_m(x, y; t) = \frac{1}{h_m} \sum_{q=1}^{\infty} 2^{-q} \sum_{\substack{\gamma \in \Gamma_m : \gamma_0 = x, \gamma_q = y \\ |\gamma_j - \gamma_{j-1}| = 1 \quad \forall j \geq 1}} W_m(\gamma, q, t)$$

where

$$(3.4) \quad W_m(\gamma, q, t) = \frac{1}{h_m} \int_0^t ds_1 \int_{s_1}^t ds_2 \dots \int_{s_{q-1}}^t ds_q e^{(t-s_q)\mathcal{L}^m(\gamma_q, \gamma_q)} \prod_{j=0}^{q-1} \left(e^{(s_{j+1}-s_j)\mathcal{L}^m(\gamma_j, \gamma_j)} 2\mathcal{L}^m(\gamma_j, \gamma_{j+1}) \right)$$

with $s_0 = 0$.

Let us introduce the following Green's function:

$$(3.5) \quad G_m(x, y; \omega) = \int_0^\infty u_m(x, y; t) e^{-i\omega t} dt = h_m^{-1} \frac{1}{\mathcal{L}^m + i\omega}(x, y).$$

The propagator can be expressed as the following contour integral

$$(3.6) \quad u_m(x, y; t) = \int_{\mathcal{C}_-} \frac{d\omega}{2\pi} G_m(x, y; \omega) e^{i\omega t} + \int_{\mathcal{C}_+} \frac{d\omega}{2\pi} G_m(x, y; \omega) e^{i\omega t}.$$

Here, \mathcal{C}_+ is the contour joining the point D to the points E, A, B in Fig. 1, while \mathcal{C}_- is the contour joining the point B to C to D . By design, each point ω on the upper path \mathcal{C}_+ is separated from the spectrum of \mathcal{L} .

Lemma 1. *There is a constant c such that, if $m \geq m_0$ then*

$$(3.7) \quad \left| \int_{\mathcal{C}_+} \frac{d\omega}{2\pi} G_m(x, y; \omega) e^{i\omega t} \right| \leq ch^2.$$

Proof. The proof is based on the geometric series expansion

$$(3.8) \quad G_m(\omega) = h_m^{-1} \frac{1}{\mathcal{L}^m + i\omega} = h_m^{-1} \sum_{j=0}^{\infty} \frac{1}{\frac{1}{2}\sigma^2\Delta^m + i\omega} \left[\mu \nabla^m \frac{1}{\frac{1}{2}\sigma^2\Delta^m + i\omega} \right]^j$$

whose convergence for $\omega \in \mathcal{C}_+$ can be established by means of a Kato-Rellich relative bound, see (Kato 1966). More precisely, for any $\alpha > 0$, one can find a $\beta > 0$ such that the operators ∇^m and Δ^m satisfy the following relative bound estimate:

$$(3.9) \quad \|\nabla^m f\|_2 \leq \alpha \|\Delta^m f\|_2 + \beta \|f\|_2.$$

for all periodic functions f and all $m \geq m_0$. This bound can be derived by observing that ∇^m and Δ^m can be diagonalized simultaneously by a Fourier transform, as done in the previous section, and by observing that for any $\alpha > 0$, one can find a $\beta > 0$ such that

$$(3.10) \quad \left| \frac{\sin h_m p}{h_m} \right| \leq \alpha \left| \frac{\cos h_m p - 1}{h_m^2} \right| + \beta$$

for all $m \geq m_0$ and all $p \in B_m$.

Under the same conditions, we also have that

$$(3.11) \quad \|\mu \nabla^m f\|_2 \leq \frac{2M\alpha}{\Sigma_0^2} \left\| \frac{1}{2}\sigma^2\Delta^m f \right\|_2 + \beta \|f\|_2.$$

Hence

$$(3.12) \quad \left\| \mu \nabla^m \frac{1}{\frac{1}{2}\sigma^2\Delta^m + i\omega} f \right\|_2 \leq \frac{2M\alpha}{\Sigma_0^2} \left\| \frac{1}{2}\sigma^2\Delta^m \frac{1}{\frac{1}{2}\sigma^2\Delta^m + i\omega} f \right\|_2 + \beta \left\| \frac{1}{\frac{1}{2}\sigma^2\Delta^m + i\omega} f \right\|_2 < 1$$

where the last inequality holds if $\omega \in \mathcal{C}_+$, if α is chosen sufficiently small and if m is large enough. In this case, the geometric series expansion in (3.8) converges in L^2 operator norm. The uniform norm of the kernel $|G_m(x, y; \omega)|$ is pointwise bounded from above by h_m^{-1} .

Since the points B and D have imaginary part equal at height $4 \frac{|\log h_m|}{t}$, the integral over the contour \mathcal{C}_+ converges also and is bounded from above by ch_m^2 in uniform norm. \square

Lemma 2. *If $q \geq \frac{e^2 \Sigma_1^2 t}{2h_m^2}$ we have that*

$$(3.13) \quad W_m(\gamma, q; t) \leq \sqrt{\frac{q}{2\pi}} \exp\left(-\frac{\Sigma_0^2 t}{2} - q\right).$$

Proof. Let us define the function

$$(3.14) \quad \phi(t) = \frac{\Sigma_1^2}{2h_m^2} e^{-\frac{\Sigma_0^2 t}{2h_m^2}} 1(t \geq 0)$$

where $1(t \geq 0)$ is the characteristic function of \mathbb{R}_+ . We have that

$$(3.15) \quad W_m(\gamma, q; t) \leq \phi^{*q}(t)$$

where ϕ^{*q} is the q -th convolution power, i.e. the q -fold convolution product of the function ϕ by itself. The Fourier transform of $\phi(t)$ is given by

$$(3.16) \quad \hat{\phi}(\omega) = \frac{\Sigma_1^2}{2h_m^2} \int_0^\infty e^{-i\omega t - \frac{\Sigma_0^2 t}{2h_m^2}} dt = \frac{\Sigma_1^2}{2i\omega h_m^2 + \Sigma_0^2}.$$

The convolution power is given by the following inverse Fourier transform:

$$(3.17) \quad \phi^{*q}(t) = \int_{-\infty}^\infty \hat{\phi}(\omega)^q e^{i\omega t} \frac{d\omega}{2\pi} = \left(\frac{\Sigma_1}{\Sigma_0}\right)^{2q} \int_{-\infty}^\infty \left(1 + \frac{2i\omega h_m^2}{\Sigma_0^2}\right)^{-q} e^{i\omega t} \frac{d\omega}{2\pi}.$$

Introducing the new variable $z = 1 + \frac{2i\omega h_m^2}{\Sigma_0^2}$, the integral can be recast as follows

$$(3.18) \quad \phi^{*q}(t) = \frac{\Sigma_0^{2-2q} \Sigma_1^{2q}}{4\pi i h_m^2} \lim_{R \rightarrow \infty} \int_{\mathcal{C}_R} z^{-q} \exp\left(\frac{\Sigma_0^2 t}{2h_m^2} (z-1)\right) dz$$

where \mathcal{C}_R is the contour in Fig. 2. Using the residue theorem and noticing that the only pole of the integrand is at $z = 0$, we find

$$(3.19) \quad \phi^{*q}(t) = \frac{1}{(q-1)!} \left(\frac{\Sigma_1^2 t}{2h_m^2}\right)^q \exp\left(\frac{-\Sigma_0^2 t}{2h_m^2}\right).$$

Making use of Stirling's formula $q! \approx \sqrt{2\pi} q^{q+\frac{1}{2}} e^{-q}$, we find

$$(3.20) \quad \phi^{*q}(t) \approx \sqrt{\frac{q}{2\pi}} \exp\left(-\frac{\Sigma_0^2 t}{2h_m^2} + q \log \frac{\Sigma_1^2 t}{2h_m^2} + q(1 - \log q)\right).$$

If $\log q \geq \log \frac{\Sigma_1^2 t}{2h_m^2} + 2$, then we arrive at the bound in (3.13). □

It suffices to consider the case $m' = m + 1$ for all values of m above a fixed threshold. In fact, given this particular case, the general statement can be derived with an iterative argument. To this end, we introduce a renormalization group transformation based on the notion of decorating path.

Definition 1. (Decorating Paths.) Let $m \geq m_0$ and let $\gamma = \{y_0, y_1, y_2, \dots\}$ be a symbolic sequence in Γ_m . A decorating path around γ is defined as a symbolic sequence $\gamma' = \{y_0, y'_1, y'_2, \dots\}$ with $y'_i \in h_{m+1}\mathbb{Z}$ containing the sequence γ as a subset and such that if $y'_j = y_i$ and $y'_k = y_{i+1}$, then all elements y'_n with $j < n < k$ are such that $|y'_n - y'_j| \leq h_{m+1}$. Let $\mathcal{D}_{m+1}(\gamma)$ be the set of all decorating sequences around γ . The decorated weights are defined as follows:

$$(3.21) \quad \tilde{W}_m(\gamma, q; t) = \sum_{q'=q}^\infty \sum_{\substack{\gamma' \in \mathcal{D}_{m+1}(\gamma) \\ \gamma'_{q'} = \gamma_q}} W_{m+1}(\gamma', q'; t).$$

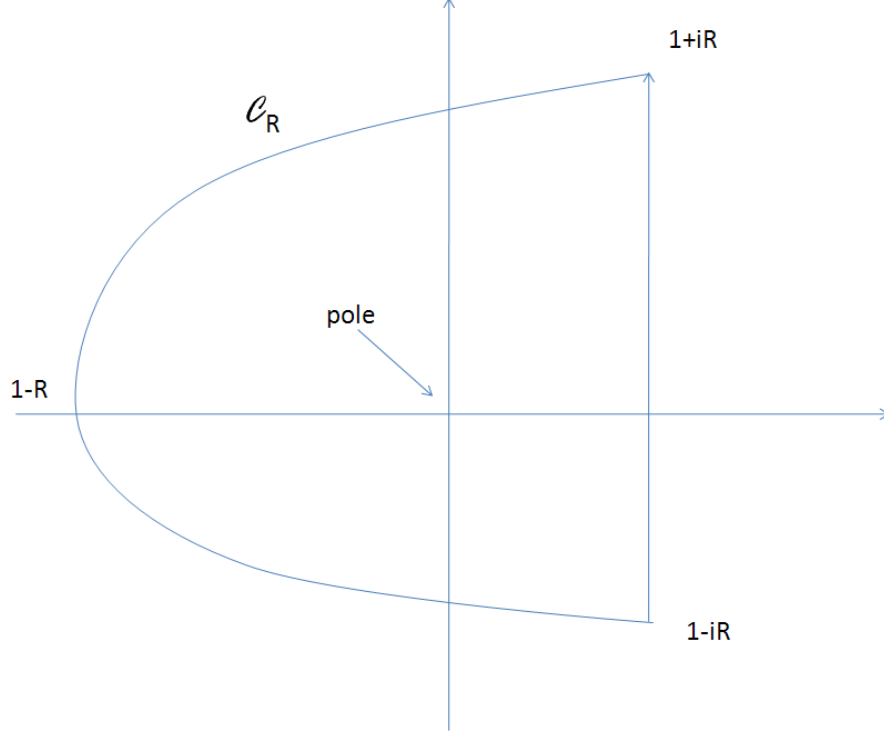
Finally, let us introduce also the following Fourier transform:

$$(3.22) \quad \hat{W}_m(\gamma, q; \omega) = \int_0^\infty W_m(\gamma, q; t) e^{i\omega t} dt, \quad \hat{\tilde{W}}_m(\gamma, q; \omega) = \int_0^\infty \tilde{W}_m(\gamma, q; t) e^{i\omega t} dt.$$

Notation 1. In the following, we set $h = h_{m+1}$ so that $h_m = 2h$. We also use the Landau notation $O(h^n)$ to indicate a function $f(h)$ such that $h^{-n} f(h)$ is bounded in a neighborhood of 0.

Lemma 3. Let $x, y \in A_m$ and let \mathcal{C}_- be an integration contour as in Fig. 1. Then

$$(3.23) \quad \left| \left(\int_{\mathcal{C}_-} 2G_{m+1}(x, y; \omega) - G_m(x, y; \omega) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right| = O(h^2).$$


 FIGURE 2. Contour of integration \mathcal{C}_R for the integral in (3.18).

Proof. We have that

$$(3.24) \quad 2G_{m+1}(x, y; \omega) - G_m(x, y; \omega) = \frac{1}{h} \sum_{q=1}^{\infty} 2^{-q} \sum_{\substack{\gamma \in \Gamma_m : \gamma_0 = x, \gamma_q = y \\ |\gamma_j - \gamma_{j-1}| = 1 \forall j \geq 1}} \left(2\hat{W}_m(\gamma, q; \omega) - \hat{W}_m(\gamma, q; \omega) \right).$$

The number of paths over which the summation is extended is

$$(3.25) \quad N(\gamma, q; x, y) \equiv \#\{\gamma \in \Gamma_m : \gamma_0 = x, \gamma_q = y, |\gamma_j - \gamma_{j-1}| = 1 \forall j \geq 1\} = \binom{q}{\frac{q}{2} + k}$$

where $k = \frac{|y-x|}{h_m}$. Applying Stirling's formula we find

$$(3.26) \quad N_\gamma \lesssim 2^q \sqrt{\frac{2}{\pi q}}.$$

Hence

$$(3.27) \quad \begin{aligned} & \left| \int_{\mathcal{C}_-} \left(2G_{m+1}(x, y; \omega) - G_m(x, y; \omega) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right| \\ & \leq \frac{c}{h} \sum_{q=1}^{\infty} \sqrt{\frac{1}{q}} \max_{\substack{\gamma \in \Gamma_m : \gamma_0 = x, \gamma_q = y \\ |\gamma_j - \gamma_{j-1}| = 1 \forall j \geq 1}} \left| \int_{\mathcal{C}_-} \left(2\hat{W}_m(\gamma, q; \omega) - \hat{W}_m(\gamma, q; \omega) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right|. \end{aligned}$$

for some constant $c \approx \sqrt{\frac{2}{\pi}} > 0$. It suffices to extend the summation over q only up to

$$(3.28) \quad q_{\max} \equiv \frac{e^2 \Sigma_1^2 t}{2h^2}.$$

To resum beyond this threshold, one can use the previous lemma. More precisely, we have that

$$(3.29) \quad \begin{aligned} & \left| \int_{\mathcal{C}_-} \left(2G_{m+1}(x, y; \omega) - G_m(x, y; \omega) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right| \\ & \leq \frac{c\sqrt{q_{\max}}}{h} \max_{\substack{q, \gamma \in \Gamma_m : \gamma_0 = x, \gamma_q = y \\ |\gamma_j - \gamma_{j-1}| = 1 \forall j \geq 1}} \left| \int_{\mathcal{C}_-} \left(2\hat{W}_m(\gamma, q; \omega) - \hat{W}_m(\gamma, q; \omega) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right|. \end{aligned}$$

Let $v(x) = \sigma(x)^2$. To evaluate the resummed weight function, let us form the matrix

$$(3.30) \quad \bar{\mathcal{L}}(x; h) = \begin{pmatrix} -\frac{v(x+h)}{h^2} & \frac{v(x+h)}{2h^2} - \frac{\mu(x+h)}{2h} & 0 \\ \frac{v(x)}{2h^2} + \frac{\mu(x)}{2h} & -\frac{v(x)}{h^2} & \frac{v(x)}{2h^2} - \frac{\mu(x)}{2h} \\ 0 & \frac{v(x-h)}{2h^2} + \frac{\mu(x-h)}{2h} & -\frac{v(x-h)}{h^2} \end{pmatrix}$$

and decompose it as follows:

$$(3.31) \quad \bar{\mathcal{L}}(x; h) = \frac{1}{h^2} \bar{\mathcal{L}}_0(x) + \frac{1}{h} \bar{\mathcal{L}}_1(x) + \bar{\mathcal{L}}_2(x) + h \bar{\mathcal{L}}_3(x) + O(h^2).$$

where

$$(3.32) \quad \bar{\mathcal{L}}_0(x) = \begin{pmatrix} -v(x) & \frac{1}{2}v(x) & 0 \\ \frac{1}{2}v(x) & -v(x) & \frac{1}{2}v(x) \\ 0 & \frac{1}{2}v(x) & -v(x) \end{pmatrix},$$

$$(3.33) \quad \bar{\mathcal{L}}_1(x) = \begin{pmatrix} -v'(x) & \frac{1}{2}v'(x) - \frac{1}{2}\mu(x) & 0 \\ \frac{1}{2}\mu(x) & 0 & -\frac{1}{2}\mu(x) \\ 0 & -\frac{1}{2}v'(x) + \frac{1}{2}\mu(x) & v'(x) \end{pmatrix},$$

$$(3.34) \quad \bar{\mathcal{L}}_2(x) = \begin{pmatrix} -\frac{1}{2}v''(x) & \frac{1}{4}v''(x) - \frac{1}{2}\mu'(x) & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{4}v''(x) - \frac{1}{2}\mu'(x) & -\frac{1}{2}v''(x) \end{pmatrix}.$$

and

$$(3.35) \quad \bar{\mathcal{L}}_3(x) = \begin{pmatrix} -\frac{1}{6}v'''(x) & \frac{1}{12}v'''(x) - \frac{1}{4}\mu''(x) & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{12}v'''(x) + \frac{1}{4}\mu''(x) & \frac{1}{6}v'''(x) \end{pmatrix}.$$

Let us introduce the sign variable $\tau = \pm 1$, the functions

$$(3.36) \quad \phi_0(t, x, \tau) \equiv 2\mathcal{L}^m(x, x + 2\tau h) e^{t\mathcal{L}^m(x, x)} 1(t \geq 0)$$

$$(3.37) \quad \phi_1(t, x, \tau) \equiv 2\mathcal{L}^{m+1}(x + \tau h, x + 2\tau h) e^{t\bar{\mathcal{L}}(x; h)}(x, x + \tau h) 1(t \geq 0)$$

and their Fourier transforms

$$(3.38) \quad \begin{aligned} \hat{\phi}_0(\omega, x, \tau) &= \left(\frac{v(x)}{4h^2} + \tau \frac{\mu(x)}{2h} \right) \left(\frac{v(x)}{4h^2} + i\omega \right)^{-1} \\ \hat{\phi}_1(\omega, x, \tau) &= \left(\frac{v(x)}{h^2} + \tau \frac{\mu(x) + v'(x)}{h} + \frac{v''(x) + \mu'(x)}{2} + \left(\frac{v'''(x)}{6} + \frac{\mu''(x)}{2} \right) \tau h + O(h^2) \right) \\ & \quad \langle x | (-\bar{\mathcal{L}}(x; h) + i\omega)^{-1} | x + \tau h \rangle. \end{aligned}$$

where

$$(3.39) \quad |x \rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad |x + \tau h \rangle = \begin{pmatrix} \delta_{\tau,1} \\ 0 \\ \delta_{\tau,-1} \end{pmatrix}.$$

We also require the functions

$$(3.40) \quad \psi_0(t, x) \equiv e^{t\mathcal{L}^m(x,x)}1(t \geq 0), \quad \psi_1(t, x) \equiv e^{t\bar{\mathcal{L}}(y;h)}(x, x)1(t \geq 0)$$

and the corresponding Fourier transforms

$$(3.41) \quad \hat{\psi}_0(\omega, x) = \left(\frac{v(x)}{4h^2} + i\omega \right)^{-1}, \quad \hat{\psi}_1(\omega, x) = \langle x | (-\bar{\mathcal{L}}(x; h) + i\omega)^{-1} | x \rangle.$$

If γ is a symbolic sequence, then

$$(3.42) \quad \hat{W}_m(\gamma, q; \omega) = \hat{\psi}_0(\omega, \gamma_q) \prod_{j=0}^{q-1} \hat{\phi}_0(\omega; \gamma_j, \text{sgn}(\gamma_{j+1} - \gamma_j))$$

$$(3.43) \quad \hat{\tilde{W}}_m(\gamma, q; \omega) = \hat{\psi}_1(\omega, \gamma_q) \prod_{j=0}^{q-1} \hat{\phi}_1(\omega; \gamma_j, \text{sgn}(\gamma_{j+1} - \gamma_j)).$$

Let us estimate the difference between the functions $\hat{\phi}_1(\omega, x, \tau)$ and $\hat{\phi}_2(\omega, x, \tau)$ assuming that ω is in the contour \mathcal{C}_- in Fig. 2. Retaining only terms up to order up to $O(h^3)$, we find

$$(3.44) \quad \hat{\phi}_0(\omega, x, \tau) = 1 + \frac{2\mu(x)\tau h}{v(x)} - \frac{4i\omega h^2}{v(x)} - 8\mu(x) \frac{i\omega\tau h^3}{v(x)^2} - \frac{16\omega^2 h^4}{v(x)^2} + O(h^5).$$

A lengthy but straightforward calculation which is best carried out using a symbolic manipulation program, gives

$$(3.45) \quad \begin{aligned} \hat{\phi}_1(\omega, x, \tau) = 1 + \frac{2\mu(x)\tau h}{v(x)} - \frac{4i\omega h^2}{v(x)} - [8\mu(x) - v'(x)] \frac{i\omega\tau h^3}{v(x)^2} \\ + r(x) \cdot h^3\tau + i\omega h^4 p(x) - \frac{14\omega^2 h^4}{v(x)^2} + O(h^5) \end{aligned}$$

where

$$(3.46) \quad \begin{aligned} r(x) &= \frac{1}{2v(x)^3} [\mu''(x)v(x)^2 - 4\mu(x)^3 + 2v'(x)\mu(x)^2 - 2v''(x)v(x)\mu'(x) \\ &\quad - (2\mu(x)\mu'(x) + v''(x)v(x) - 2v'(x)^2)\mu(x)]. \\ p(x) &= \frac{1}{v(x)^3} [-4\mu(x)^2 + 2v'(x)\mu(x) - 4v(x)\mu'(x) - v''(x)v(x) + 2v'(x)^2]. \end{aligned}$$

We have that

$$(3.47) \quad \begin{aligned} &\sum_{j=0}^{q-1} \left(\log \hat{\phi}_0(\omega; \gamma_j, \text{sgn}(\gamma_{j+1} - \gamma_j)) - \log \hat{\phi}_1(\omega; \gamma_j, \text{sgn}(\gamma_{j+1} - \gamma_j)) \right) \\ &= \sum_{j=0}^{q-1} \left(\frac{i\omega v'(\gamma_j)}{v(\gamma_j)^2} + r(\gamma_j) \right) h^3 \text{sgn}(\gamma_{j+1} - \gamma_j) + (|\omega| \|p\|_\infty + 2|\omega|^2 \|v^{-2}\|_\infty) O(h^4 q) \\ &= i\omega h^2 \left(\frac{1}{v(\gamma_0)} - \frac{1}{v(\gamma_q)} \right) + h^2 (R(\gamma_q) - R(\gamma_0)) + (|\omega| \|p\|_\infty + 2|\omega|^2 \|v^{-2}\|_\infty) O(h^4 q) \end{aligned}$$

where $R(x)$ is a primitive of $r(x)$, i.e.

$$(3.48) \quad R(x) = \int_{L_x}^x r(x') dx'.$$

We conclude that there is a constant $c > 0$ such that

$$(3.49) \quad \left| \int_{\mathcal{C}_-} \left(\prod_{j=0}^{q-1} \hat{\phi}_0(\omega; \gamma_j, \text{sgn}(\gamma_{j+1} - \gamma_j)) - \prod_{j=0}^{q-1} \hat{\phi}_1(\omega; \gamma_j, \text{sgn}(\gamma_{j+1} - \gamma_j)) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right| \leq ch^2.$$

for all $q \leq q_{\max}$. Here we use the decay of $e^{i\omega t}$ in the upper half of the complex ω plane to offset the ω dependencies in the integrand. Similar calculations lead to the following expansions:

$$(3.50) \quad \hat{\psi}_0(\omega, x) = \frac{4h^2}{v(x)} + O(\omega h^4), \quad \hat{\psi}_1(\omega, x) = \frac{2h^2}{v(x)} + O(\omega h^4) = \frac{1}{2} \hat{\psi}_0(\omega, x) + O(\omega h^4).$$

Since $q < ch^{-2}$ and $\omega \leq |\log h|$, we find

$$(3.51) \quad \left| \int_{\mathcal{C}_-} \left(2G_{m+1}(x, y; \omega) - G_m(x, y; \omega) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right| \leq c \frac{\sqrt{q_{\max}}}{h} h^4 \leq ch^2.$$

By differentiating with respect to time in equation (3.52), we find that

$$(3.52) \quad \frac{\partial}{\partial t} u_m(x, y; t) = \int_{\mathcal{C}_-} i\omega G_m(x, y; \omega) e^{i\omega t} \frac{d\omega}{2\pi} + \int_{\mathcal{C}_+} i\omega G_m(x, y; \omega) e^{i\omega t} \frac{d\omega}{2\pi}.$$

All the derivations above carry through and we conclude that

$$(3.53) \quad \left| \int_{\mathcal{C}_+} \frac{d\omega}{2\pi} i\omega G_m(x, y; \omega) e^{i\omega t} \right| \leq ch^2.$$

and also

$$(3.54) \quad \left| \int_{\mathcal{C}_-} i\omega \left(2G_{m+1}(x, y; \omega) - G_m(x, y; \omega) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right| \leq c \frac{\sqrt{q_{\max}}}{h} h^4 \leq ch^2.$$

Hence the first time derivatives of the kernel satisfy a similar Cauchy convergence condition as the kernel itself. \square

4. LESSER SMOOTH COEFFICIENTS

In this section we assume coefficients are either Hölder continuous or obey the conditions in Theorem 2.

Lemma 4. *Let $f(x)$ be a continuous function in $[-L, L]$ satisfying periodic boundary conditions. Then, for all $h > 0$, we have that*

$$(4.1) \quad f(x + h_m) = f(x) + h_m \nabla_x^m f(x) + \frac{h^2}{2} \Delta_x^m f(x)$$

$$(4.2) \quad f(x - h_m) = f(x) - h_m \nabla_x^m f(x) + \frac{h^2}{2} \Delta_x^m f(x).$$

This is the result of a simple calculation, which is however useful as it allows one to extend the derivation in the previous section by making the following replacements:

$$(4.3) \quad v'(x) \rightarrow \nabla_x^m v(x), \quad v''(x) \rightarrow \Delta_x^m v(x), \quad v'''(x) \rightarrow 0$$

$$(4.4) \quad \mu'(x) \rightarrow \nabla_x^m \mu(x), \quad \mu''(x) \rightarrow \Delta_x^m \mu(x).$$

In fact,

$$(4.5) \quad \bar{\mathcal{L}}(x; h) = \frac{1}{h^2} \bar{\mathcal{L}}_0(x) + \frac{1}{h} \bar{\mathcal{L}}_1(x) + \bar{\mathcal{L}}_2(x) + h \bar{\mathcal{L}}_3(x)$$

without any $O(h^3)$ corrections as long as one re-defines the matrices on the right hand side as follows:

$$(4.6) \quad \bar{\mathcal{L}}_0(x) = \begin{pmatrix} -v(x) & \frac{1}{2}v(x) & 0 \\ \frac{1}{2}v(x) & -v(x) & \frac{1}{2}v(x) \\ 0 & \frac{1}{2}v(x) & -v(x) \end{pmatrix},$$

$$(4.7) \quad \bar{\mathcal{L}}_1(x) = \begin{pmatrix} -\nabla_x^m v(x) & \frac{1}{2}\nabla_x^m v(x) - \frac{1}{2}\mu(x) & 0 \\ \frac{1}{2}\mu(x) & 0 & -\frac{1}{2}\mu(x) \\ 0 & -\frac{1}{2}\nabla_x^m v(x) + \frac{1}{2}\mu(x) & \nabla_x^m v(x) \end{pmatrix},$$

$$(4.8) \quad \bar{\mathcal{L}}_2(x) = \begin{pmatrix} -\frac{1}{2}\Delta_x^m v(x) & \frac{1}{4}\Delta_x^m v(x) - \frac{1}{2}\nabla_x^m \mu(x) & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{4}\Delta_x^m v(x) - \frac{1}{2}\nabla_x^m \mu(x) & -\frac{1}{2}\Delta_x^m v(x) \end{pmatrix}.$$

and

$$(4.9) \quad \bar{\mathcal{L}}_3(x) = \begin{pmatrix} 0 & -\frac{1}{4}\Delta_x^m \mu(x) & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{4}\Delta_x^m \mu(x) & 0 \end{pmatrix}.$$

All derivations in the previous section go through formally unchanged and one arrives at the following expressions

$$(4.10) \quad \hat{\phi}_0(\omega, x, \tau) = 1 + \frac{2\mu(x)\tau h}{v(x)} - \frac{4i\omega h^2}{v(x)} - 8\mu(x)\frac{i\omega\tau h^3}{v(x)^2} - \frac{16\omega^2 h^4}{v(x)^2} + O(h^5).$$

and

$$(4.11) \quad \begin{aligned} \hat{\phi}_1(\omega, x, \tau) = 1 + \frac{2\mu(x)\tau h}{v(x)} - \frac{4i\omega h^2}{v(x)} - [8\mu(x) - \nabla_x^m v(x)]\frac{i\omega\tau h^3}{v(x)^2} \\ + r(x) \cdot h^3\tau + i\omega h^4 p(x) - \frac{14\omega^2 h^4}{v(x)^2} + O(h^5) \end{aligned}$$

where

$$(4.12) \quad \begin{aligned} r(x) &= \frac{1}{2v(x)^3} \left[v(x)^2 \Delta_x^m m(x) - 4m(x)^3 + 2\nabla_x^m v(x)m(x)^2 - 2v(x)\nabla_x^m v(x)\nabla_x^m m(x) \right. \\ &\quad \left. - (2m(x)\nabla_x^m m(x) + v(x)\Delta_x^m v(x) - 2(\nabla_x^m v(x))^2)m(x) \right]. \\ p(x) &= \frac{1}{v(x)^3} \left[-4m(x)^2 + 2m(x)\nabla_x^m v(x) - 4v(x)\nabla_x m(x) - v(x)\Delta_x^m v(x) + 2(\nabla_x^m v(x))^2 \right]. \end{aligned}$$

We have that

$$(4.13) \quad \begin{aligned} &\left| \sum_{j=0}^{q-1} \left(\log \hat{\phi}_0(\omega; \gamma_j, \text{sgn}(\gamma_{j+1} - \gamma_j)) - \log \hat{\phi}_1(\omega; \gamma_j, \text{sgn}(\gamma_{j+1} - \gamma_j)) \right) \right| \\ &\leq \left| \sum_{j=0}^{q-1} \left(\frac{i\omega \nabla_{\gamma_j}^m v(\gamma_j)}{v(\gamma_j)^2} + r(\gamma_j) \right) h^3 \text{sgn}(\gamma_{j+1} - \gamma_j) \right| + (|\omega| \|p\|_\infty + 2|\omega|^2 \|v^{-2}\|_\infty) O(h^4 q). \\ &\leq 2Lh^2 \sup_{x \in A_m} \left| \frac{i\omega \nabla_x^m v(x)}{v(x)^2} + r(x) \right| + (|\omega| \|p\|_\infty + 2|\omega|^2 \|v^{-2}\|_\infty) O(h^4 q) \leq ch^\gamma. \end{aligned}$$

where in the last step we made use of the Hölder continuity assumptions of Theorem 1. The other bounds staying the same, we arrive at

$$(4.14) \quad \left| \int_{\mathcal{C}_-} \left(2G_{m+1}(x, y; \omega) - G_m(x, y; \omega) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right| \leq c \frac{\sqrt{q_{\max}}}{h} h^{2+\gamma} \leq ch^\gamma.$$

Under the weaker assumption of Theorem 2, the bound that applies is instead

$$(4.15) \quad \left| \int_{\mathcal{C}_-} \left(2G_{m+1}(x, y; \omega) - G_m(x, y; \omega) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right| \leq c \frac{\sqrt{q_{\max}}}{h} h^2 \rho(h) \leq c\rho(h).$$

Similar bounds also extend to the case of the first time derivative, since multiplication by a factor $i\omega$ inside of the contour integral is immaterial as far as establishing a bound of this sort is concerned. This completes the proof of Theorem 1 and Theorem 2.

5. EXPLICIT EULER SCHEME

In this section we prove Theorem 3. A Dyson expansion can also be obtained for the time-discretized kernel and has the form

$$(5.1) \quad u_m^{\delta t}(y_1, y_2; t) = \frac{1}{h_m} \sum_{q=1}^{\infty} \sum_{\gamma \in \Gamma_m: \gamma_0=y_1, \gamma_q=y_2} \sum_{k_1=1}^N \sum_{k_2=k_1+1}^N \dots \sum_{k_q=k_{q-1}+1}^N \left(1 + \delta t \mathcal{L}_m(\gamma_0, \gamma_0) \right)^{k_1-1} (\delta t)^q \prod_{j=1}^q \mathcal{L}_m(\gamma_{j-1}, \gamma_j) \left(1 + \delta t \mathcal{L}_m(\gamma_j, \gamma_j) \right)^{k_{j+1}-k_j-1}$$

where $t_{q+1} = t$ and $k_{q+1} = N$. In this case, the propagator can be expressed through a Fourier integral as follows:

$$(5.2) \quad u_m^{\delta t}(y_1, y_2; t) = \int_{-\frac{\pi}{\delta t}}^{\frac{\pi}{\delta t}} G_m^{\delta t}(y_1, y_2; \omega) e^{i\omega t} \frac{d\omega}{2\pi}$$

where

$$(5.3) \quad G_m^{\delta t}(y_1, y_2; \omega) = \delta t \sum_{j=0}^{\infty} u_m^{\delta t}(y_1, y_2; j\delta t) e^{-i\omega j\delta t}.$$

The propagator can also be represented as the limit

$$(5.4) \quad u_m^{\delta t}(y_1, y_2; t) = \lim_{H \rightarrow \infty} \int_{\mathcal{C}_H} G_m^{\delta t}(y_1, y_2; \omega) e^{i\omega t} \frac{d\omega}{2\pi}$$

where \mathcal{C}_H is the contour in Fig. 3. This is due to the fact that the integral along the segments BC and DA are the negative of each other, while the integral over CD tends to zero exponentially fast as $\Im(\omega) \rightarrow \infty$, where $\Im(\omega)$ is the imaginary part of ω . Using Cauchy's theorem, the contour in Fig. 3 can be deformed into the contour in Fig. 1. To estimate the discrepancy between the time-discretized kernel and the continuous time one, one can thus compare the Green's function along such contour. Again, the only arc that requires detailed attention is the arc BCD , as the integral over rest of the contour of integration can be bounded from above as in the previous section.

Let $h = h_m$ and let us introduce the two functions

$$(5.5) \quad \phi_0(t, x, \tau) \equiv 2\mathcal{L}_m(x, x + \tau h) e^{t\mathcal{L}_m(x, x)} 1(t \geq 0),$$

$$(5.6) \quad \phi_{\delta t}(j, x, \tau) \equiv 2\mathcal{L}_m(x, x + \tau h) \left(1 + \delta t \mathcal{L}_m(x, x) \right)^{j-1}.$$

and the corresponding Fourier transforms

$$(5.7) \quad \hat{\phi}_0(\omega, x, \tau) = \int_0^\infty \phi_0(t, x, \tau) e^{-i\omega t} \frac{d\omega}{2\pi} = \left(\frac{v(x)}{h^2} + \tau \frac{\mu(x)}{h} \right) \left(\frac{v(x)}{h^2} + i\omega \right)^{-1}$$

$$(5.8) \quad \hat{\phi}_{\delta t}(\omega, x, \tau) = \sum_{j=0}^{\frac{\delta t}{h}} \phi_{\delta t}(j, x, \tau) e^{-i\omega j\delta t} = \left(\frac{v(x)}{h^2} + \tau \frac{\mu(x)}{h} \right) \left(e^{i\omega \delta t} - 1 + \delta t \frac{v(x)}{h^2} \right)^{-1}.$$

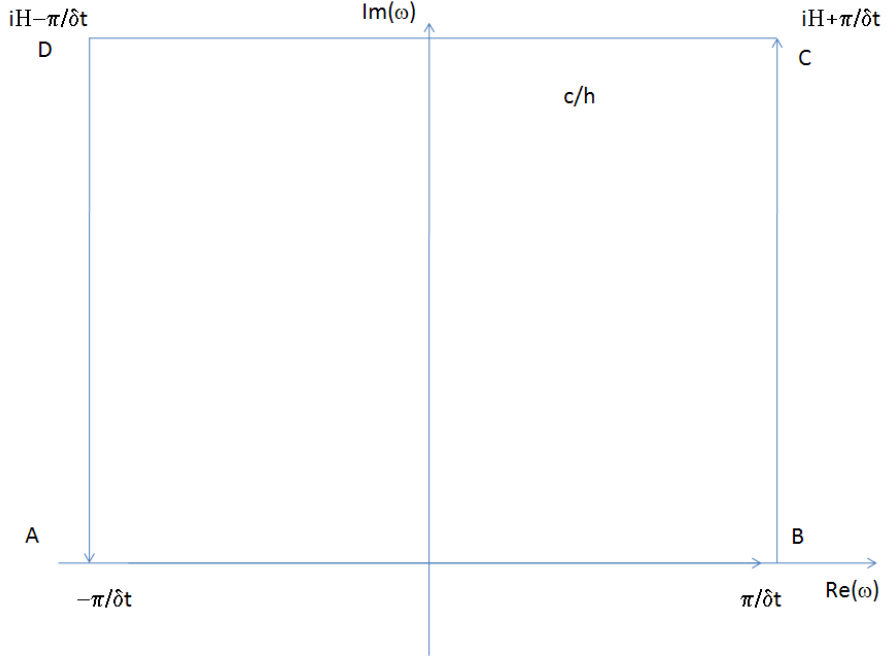


FIGURE 3. Contour of integration for the integral in (5.4).

We have that

$$\begin{aligned}
 \hat{\phi}_{\delta t}(\omega, x, \tau) &= \left(\frac{v(x)}{h^2} + \tau \frac{\mu(x)}{h} \right) \left(i\omega + \frac{v(x)}{h^2} - \frac{\omega^2}{2} \delta t + O(\delta t^2) \right)^{-1} \\
 (5.9) \quad &= \hat{\phi}_0(\omega, x, \tau) + \frac{\omega^2}{2v(x)} h^2 \delta t + O(h^2 \delta t^2). = \hat{\phi}_0(\omega, x, \tau) + O(h^4),
 \end{aligned}$$

where the last step uses the fact that $\delta t = O(h^2)$.

Let us also introduce the functions

$$(5.10) \quad \psi_0(t, x, \tau) \equiv e^{t\mathcal{L}_m(x, x)} \mathbf{1}(t \geq 0), \quad \psi_{\delta t}(j, x, \tau) \equiv \sum_{k=1}^j (1 + \delta t \mathcal{L}_m(x, x))^{j-1}.$$

and the corresponding Fourier transforms

$$(5.11) \quad \hat{\psi}_0(\omega, x, \tau) = \left(\frac{v(x)}{h^2} + i\omega \right)^{-1}, \quad \hat{\psi}_{\delta t}(\omega, x, \tau) = \left(e^{i\omega \delta t} - 1 + \delta t \frac{v(x)}{h^2} \right)^{-1}.$$

Again we find that

$$(5.12) \quad \hat{\psi}_0(\omega, x, \tau) = \hat{\psi}_{\delta t}(\omega, x, \tau) + O(h^4).$$

If γ is a symbolic sequence, then let us set

$$(5.13) \quad \hat{W}_m(\gamma, q; \omega) = \hat{\psi}_0(\omega, \gamma_q) \prod_{j=0}^{q-1} \hat{\phi}_0(\omega; \gamma_j, \text{sgn}(\gamma_{j+1} - \gamma_j))$$

$$(5.14) \quad \hat{W}_m^{\delta t}(\gamma, q; \omega) = \hat{\psi}_{\delta t}(\omega, \gamma_q) \prod_{j=0}^{q-1} \hat{\phi}_{\delta t}(\omega; \gamma_j, \text{sgn}(\gamma_{j+1} - \gamma_j)).$$

We have that

$$(5.15) \quad G_m^{\delta t}(x, y; \omega) - G_m(x, y; \omega) = \frac{1}{h} \sum_{q=1}^{\infty} 2^{-q} \sum_{\substack{\gamma \in \Gamma_m : \gamma_0 = x, \gamma_q = y \\ |\gamma_j - \gamma_{j-1}| = 1 \forall j \geq 1}} \left(\hat{W}_m^{\delta t}(\gamma, q; \omega) - \hat{W}_m(\gamma, q; \omega) \right).$$

The integration over the contour in Fig. 1 can again be split into an integration over the contour \mathcal{C}_- and an integration over \mathcal{C}_+ . The integral over \mathcal{C}_+ can be bounded from above thanks to Lemma 1. Furthermore, we have that

$$(5.16) \quad \left| \int_{\mathcal{C}_-} \left(G_m^{\delta t}(x, y; \omega) - G_m(x, y; \omega) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right| \leq ch^{-1} \sqrt{q_{\max}} \max_{\substack{q, \gamma \in \Gamma_m : \gamma_0 = x, \gamma_q = y \\ |\gamma_j - \gamma_{j-1}| = 1 \forall j \geq 1}} \left| \int_{\mathcal{C}_-} \left(\hat{W}_m^{\delta t}(\gamma, q; \omega) - \hat{W}_m(\gamma, q; \omega) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right| \leq ch^2$$

To bound the time derivative, we have to consider

$$(5.17) \quad \left| \int_{\mathcal{C}_-} \left(\frac{e^{i\omega \delta t} - 1}{\delta t} G_m^{\delta t}(x, y; \omega) - i\omega G_m(x, y; \omega) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right|$$

But, since $\delta t = O(h^2)$, also this difference is $O(h^2)$.

6. CONCLUSIONS

We obtained bounds on convergence rates for explicit discretization schemes to the kernel of one-dimensional diffusion equations with continuous coefficients. We consider both semidiscrete triangulations with continuous time and explicit Euler schemes with time step small enough for the method to be stable. The proof is constructive and based on a new technique of path conditioning for Markov chains and a renormalization group argument. Convergence rates depend on the degree of smoothness and Hölder differentiability of the coefficients. The method is of more general applicability and will be extended in future work.

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