STOCHASTIC INTEGRALS AND ABELIAN PROCESSES

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Abstract. We study triangulation schemes for the joint kernel of a diffusion process with uniformly continuous coefficients and an adapted, non-resonant Abelian process. The prototypical example of Abelian process to which our methods apply is given by stochastic integrals with uniformly continuous coefficients. The range of applicability includes also a broader class of processes of practical relevance, such as the sup process and certain discrete time summations we discuss.

We discretize the space coordinate in uniform steps and assume that time is either continuous or finely discretized as in a fully explicit Euler method and the Courant condition is satisfied. We show that the Fourier transform of the joint kernel of a diffusion and a stochastic integral converges in a uniform graph norm associated to the Markov generator. Convergence also implies smoothness properties for the Fourier transform of the joint kernel. Stochastic integrals are straightforward to define for finite triangulations and the convergence result gives a new and entirely constructive way of defining stochastic integrals in the continuum. The method relies on a reinterpretation and extension of the classic theorems by Feynman-Kac, Girsanov, Ito and Cameron-Martin, which are also reobtained.

We make use of a path-wise analysis without relying on a probabilistic interpretation. The Fourier representation is needed to regularize the hypo-elliptic character of the joint process of a diffusion and an adapted stochastic integral. The argument extends as long as the Fourier analysis framework can be generalized. This condition leads to the notion of non-resonant Abelian process.

Contents

1. Introduction

Consider a diffusion defined by the Markov generator

$$\mathcal{L}_x^0 = \frac{1}{2}\sigma(x)^2 \frac{\partial^2}{\partial x^2} + \mu(x)\frac{\partial}{\partial x}. \tag{1.1}$$

on the bounded interval $X = [-L_x, L_x] \subset \mathbb{R}$ where $0 < L_x < \infty$. For simplicity, we assume periodic boundary conditions and identify the two boundary points $\pm L_x$ with each other. The coefficients $\sigma(x)^2$ and $\mu(x)$ are assumed to be at least uniformly continuous and our results will depend on the degree of smoothness. We also assume that $\sigma(x) > \Sigma_0$ for some constant $\Sigma_0 > 0$.

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In this paper, we consider path dependent processes such as stochastic integrals of the form

$$y_t = \int_0^t a(x_s)dx_s + b(x_s)ds \tag{1.2}$$

where $a(x)$ and $b(x)$ are functions on $A = [-L_x, L_x]$. The case $a(x) = 0$ was considered in (Girsanov 1960), (Cameron and Martin 1949), (Feynman 1948) and (Kac 1948). The case with $a(x) \neq 0$ was first tackled in (Ito 1949). The proofs were streamlined by using martingale theory in (Doob 1953). Other references for stochastic integrals are (Kunita and Watanabe 1967), (McKean 1969) and (Meyer 1974). Most of these results assume smooth coefficients. The more general case of uniformly continuous coefficients was considered in (Stroock and Varadhan 1969).

We are interested in convergence properties of triangulation schemes for the joint distribution between the underlying process x_t and the stochastic integral y_t. If the diffusion process is approximated by a sequence of Markov chains then existence and uniqueness of stochastic integrals is not problematic. The problem is to determine conditions under which joint distributions between stochastic integrals and the underlying process converge in some meaningful norm in the limit as the triangulation becomes finer and finer, approaching the continuum limit. We establish convergence assuming uniform continuity in the coefficients of the diffusion and the stochastic integral and find convergence rates. Convergence takes place in the graph-norm of the Fourier transformed Markov generator. In the continuum limit, the Fourier transform of the joint distribution is entire analytic in the conjugate variable of the stochastic integral and is in the domain of all the powers of the Fourier transformed Markov generator. Consideration of the Fourier transform is essential in the proof as the joint distribution itself can be singular even in trivial special cases such as the one where $a(x) = b(x) = 0$. The Fourier analysis also leads to a new derivation of the celebrated formulas for characteristic functions of stochastic integrals in (Girsanov 1960), (Cameron and Martin 1949), (Feynman 1948) and (Ito 1949). Furthermore, analyticity implies convergence of moment formulas.

These results are classic and well established in the continuum limit. One of the new facets of our derivation is that it is entirely constructive and thus instructive from a computational viewpoint. We make no use of compactness arguments and non-constructive measure theory methods. The derivation is based on pathwise analysis and a renormalization group transformation. The construction applies to discretized operators and does not rely on the existence of a probabilistic interpretation. The proof of convergence is actually carried out for analytic extensions of the Fourier transform of the joint distribution, so we consider complex valued weights for the paths. The argument would also extend to different type of diffusions with complex coefficients as they occur for instance in quantum mechanics. Furthermore results are extended to discrete time approximations with fully explicit Euler schemes, in which case we also derive convergence bounds in the graph-norm for Fourier transformed joint kernels.

In (Albanese 2007*c*) we discuss the case of the kernel of diffusion processes, a more elementary situation. This paper is strictly more general and it is self-contained with no dependencies on our previous work as all derivations need to be adapted in detail. The key technique of pathwise analysis and renormalization group resummations over decorated paths are however quite similar.

Our operator algebraic approach to the problem also points out that stochastic integrals are not the most general class of path dependent processes to which these methods apply. This is based on the recognition that Fourier transforms play a key role as they allow one to block-diagonalize the joint kernel. But block-diagonalization can be achieved under more general conditions. We consider a generic situation of a diffusion and an adapted process. We associate an operator algebra to a finite triangulation of the joint process. If this operator algebra is Abelian, i.e. commutative, and if it satisfies a certain non-resonance condition, then the joint Markov generator can also be block-diagonalized and methods can be extended.

Non-resonant Abelian processes are a broad class of hypo-elliptic multidimensional diffusion processes which are computable by robust methodologies based on block-diagonalizations and fully explicit Euler schemes. Several instances of these processes found engineering applications already, see (Albanese and Vidler 2007), (Albanese and Trovato 2005), (Albanese 2007*a*), (Albanese *et al.* 2006), (Albanese and Osseiran 2007). As an example, we discuss here the case of the sup processes of a one-dimensional diffusion, i.e. of

$$y_t = \sup_{\in[0,t]} x_s. \tag{1.3}$$

We also discuss discrete time summations of the form

$$y_{T_n} = \sum_{i=1}^{n} \phi(x_{T_{i-1}}, x_{T_i}) \tag{1.4}$$

where $T_i = i\Delta T$.

The kernel convergence in graph norm we establish in this paper has noteworthy implications from the computer science viewpoint. As we discuss in (Albanese 2007*b*), they imply that explicit schemes give a robust valuation methodology for joint kernels of processes specified semi-parametrically. The Courant-Friedrichs-Lewi stability condition (Courant *et al.* 1928) for explicit methods is not much of an impediment in the presence of sufficient system memory as linear fast exponentiation can be used to accelerate the scheme. The ability of evaluating kernels robustly is the main reason why analytically solvable models are interesting and broadly used. But not all models solvable in closed form are practically computable in the sense that they allow for robust kernel valuations within the limits of double precision floating point arithmetics. Higher order hypergeometric functions are often difficult to handle and may require multiple precision. Linear fast exponentiation is a technique based on accelerated fully explicit schemes that empirically is observed to work well even in single precision and for very general model specifications. Empirically, we find that it performs better in single precision on some solvable models than the use of closed form solutions in double precision. This paper was prompted by the observation of this phenomenon and the desire to understand and explain the underlying smoothing mechanisms which is of technological significance.

The paper is organized as follows: In Section 2 we introduce notations and state our results regarding stochastic integrals. Proofs in the case of stochastic integrals are in Section 3, where we consider the case of continuous time and in Section 4 where we discuss convergence for fully explicit Euler schemes. The Ito representation is in Section 5. Extensions to the sup and other Abelian processes are given in Section 6 and conclusions end the paper.

2. Stochastic Integrals

Consider a family of increasingly fine triangulations schemes whereby the space dimension is discretized in multiples of an elementary space step $h_{xm} = L_x 2^{-m}, m \in \mathbb{N}$ and we are interested in the limit as $m \to \infty$. Let $X_m = h_{xm}\mathbb{Z} \cap X$ and consider the sequence of operators

$$\mathcal{L}_x^m = \frac{\sigma(x)^2}{2}\Delta_x^m + \mu(x)\nabla_x^m. \tag{2.1}$$

defined on the 2^{m+1}-dimensional space of all periodic functions $f_m : X_m \to \mathbb{R}$, where

$$\nabla_x^m f(x) = \frac{f(x+h_{xm}) - f(x-h_{xm})}{2h_{xm}}. \tag{2.2}$$

and

$$\Delta_x^m f(x) = \frac{f(x+h_{xm}) + f(x-h_{xm}) - 2f(x)}{{h_{xm}}^2} \tag{2.3}$$

These definitions also apply to the boundary points by periodicity. We assume that $m \geq m_0$ where m_0 is the least integer such that

$$\frac{\sigma^2(x)}{2h_{xm}^2} > \frac{|\mu(x)|}{2h_{xm}} \tag{2.4}$$

for all $m \geq m_0$ and all $x \in X_m$.

Consider the kernel $u_m(x, x'; t)$ of equation (2.6), i.e. the solution of the (forward) equation

$$\frac{\partial}{\partial t} u_m(x, x'; t) = \mathcal{L}_{x'}^{m*} u_m(x, x'; t) \tag{2.5}$$

where the operator $\mathcal{L}^{m*}_{x'}$ acts on the x' coordinate and the following initial time condition is satisfied:

$$u_m(x, x'; 0) = h_{xm}^{-1} \delta_{X_m}(x - x'). \tag{2.6}$$

where

$$\delta_{X_m}(x - x') = \begin{cases} 1 & \text{if } x = x' \mod 2L_x \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

If $f(x, x') : X_m \times X_m \to \mathbb{R}$, consider the uniform norm

$$||f_m||_{m,\infty} = \sup_{x,x' \in X_m} |f(x, x')| \tag{2.8}$$

and the graph norm

$$||f||_{m,\mathcal{L}} = ||f||_\infty + ||\mathcal{L}^m_x f||_\infty + ||\mathcal{L}^{m*}_{x'} f||_\infty. \tag{2.9}$$

In (Albanese 2007c) we show that if the coefficients $\sigma(x)^2$ and $\mu(x)$ are uniformly continuous, then the sequence of kernels u_m is Cauchy with respect to the graph norm. If in addition coefficients are Hölder continuous and $\sigma^2 \in \mathcal{C}^{k_\sigma, \alpha_\sigma}$, $\mu \in \mathcal{C}^{k_\mu, \alpha_\mu}$ and

$$\gamma \equiv \min\{2, k_\sigma + \alpha_\sigma, k_\mu + \alpha_\mu\} > 0 \tag{2.10}$$

then

$$||u_m(t) - u_{m'}(t)||_{m,\mathcal{L}} \leq c h_{xm}^\gamma \tag{2.11}$$

for all $m' > m \geq m_0$. A similar bound also holds for the kernels obtained with a fully explicit Euler scheme

$$u_m^{\delta t}(x, x'; t) = h_{xm}^{-1} \left(1 + \delta t_m \mathcal{L}^m\right)^{\left[\frac{t}{\delta t_m}\right]} (x, x'; t). \tag{2.12}$$

where δt_m is so small that

$$\min_{x \in X_m} 1 + \delta t_m \mathcal{L}^m(x, x) > 0. \tag{2.13}$$

In fact, we have that

$$||u_m(t) - u_{m'}^{\delta t}(t)||_{m,\mathcal{L}} \leq c h_{xm}^2 \tag{2.14}$$

for some constant $c > 0$.

Let h_{yn} be a monotonously descreasing sequence such as $h_{yn} \to 0$ as $n \to \infty$. Also let L_{yn} be a monotonously increasing sequence such that $L_{yn} \to \infty$ as $n \to \infty$. Let

$$Y_n = \big(h_{yn}\mathbb{Z}\big) \cap [-L_{yn}, L_{yn}]. \tag{2.15}$$

We assume again periodic boundary conditions in the y direction and identify the two extreme points of Y_n. Let's consider two-dimensional processes described by a sequence of Markov generators of the form

$$\mathcal{L}^{m,n}(x, y; x', y') = \mathcal{L}^m(x, x') + \mathcal{Q}^{m,n}(x, y; x', y') \tag{2.16}$$

where $x, x' \in X_n$ and $y, y' \in Y_n$. We assume that

$$\sum_{y' \in Y_n} \mathcal{Q}^{m,n}(x, y; x', y') = 0 \tag{2.17}$$

for all values of x, y, x', so that the marginals with respect to the first process are the same as under the dynamics given by $\mathcal{L}^m$, i.e.

$$\sum_{y' \in Y_n} \exp\big(t\mathcal{L}^{m,n}\big)(x, y; x', y') = \exp\big(t\mathcal{L}^m\big)(x; x') \tag{2.18}$$

for all triples x, y, x'. We are interested in the joint kernel

$$u_{mn}(x, y; x', y'; t) \equiv \frac{1}{h_{xm} h_{yn}} \exp\big(t\mathcal{L}^{m,n}\big)(x, y; x', y') \tag{2.19}$$

and its convergence properties in the limit as $m, n \to \infty$.

The case of general multi-dimensional diffusion processes is discussed in (Albanese and Jones 2007*a*). Results in that paper however do not apply to the class of processes in which we are interested here. In this article, we consider the case of path dependent processes which are not driven by their own diffusive dynamics and whose generator is not strongly elliptic, but rather hypoelliptic. Because of this reason, the joint kernel may be singular and a separate treatment is required.

Definition 1. *Consider a sequence of Markov generators of the form in (2.16) describing a bivariate process. One says that the second process is Abelian with respect to the first if the matrices* $Q^{m,n}_{x,x'}$ *of elements*

$$Q^{m,n}_{x,x'}(y,y') = \mathcal{Q}^{m,n}(x,y;x',y') \tag{2.20}$$

are mutually commuting, i.e. if

$$[Q^{m,n}_{x,x'}, Q^{m,n}_{x'',x'''}] = Q^{m,n}_{x,x'}Q^{m,n}_{x'',x'''} - Q^{m,n}_{x'',x'''}Q^{m,n}_{x,x'} = 0 \tag{2.21}$$

for all $x, x', x'', x''' \in X_m$.

Definition 2. *An Abelian process is called non-resonant if there is a sequence of non-singular transformations* $\mathcal{V}_{mn}(p,y)$ *which diagonalizes simultaneously the kernels* $Q^{m,n}_{x,x'}$ *for all pairs* $x, x' \in X_m$*, i.e. if*

$$\sum_{y,y' \in Y_n} \mathcal{V}_{mn}(p,y)\mathcal{Q}^{m,n}(x,y;x',y')\mathcal{V}^{-1}_{mn}(p',y') = \Lambda_{mn}(x,x')\delta_{p,p'}. \tag{2.22}$$

for all $x, x' \in X_m$*. The index* p *ranges on a set denoted with* $\hat{Y}_n$ *which has the same cardinality as* Y_n *and is called the inverse lattice of* Y_n.

In Sections 3, 4 and 5, we consider in detail the prototypical example of Abelian process, namely stochastic integrals. In this case, a triangulation for the joint generator is given by

$$\begin{aligned}\mathcal{L}^{m,n}(x,y;x',y') = &\left(\frac{\sigma(x)^2}{2h_{xm}^2} + \frac{\mu(x)}{2h_{xm}}\right)\delta_{X_m}(x'-x-h_{xm})\delta_{Y_n}\left(y'-y-\left[\frac{a(x)h_{xm}}{h_{yn}}\right]h_{yn}\right)\\ &+\left(\frac{\sigma(x)^2}{2h_{xm}^2} - \frac{\mu(x)}{2h_{xm}}\right)\delta_{X_m}(x'-x+h_{xm})\delta_{Y_n}\left(y'-y+\left[\frac{a(x)h_{xm}}{h_{yn}}\right]h_{yn}\right)\\ &-\frac{\sigma(x)^2}{h_{xm}^2}\delta_{X_m}(x'-x)\delta_{Y_n}(y'-y) + b(x)_+\delta_{xx'}\nabla^{n+}_y(y,y')\\ &+ b(x)_-\delta_{xx'}\nabla^{n-}_y(y,y')\end{aligned} \tag{2.23}$$

where

$$\nabla^{n\pm}_y g(y) = \frac{g(y\pm h_{yn}) - g(y)}{2h_{yn}}, \tag{2.24}$$

$a_\pm = \max(\pm a, 0)$ and

$$\delta_{Y_n}(y-y') = \begin{cases}1 & \text{if } \ y = y' \mod 2L_y\\ 0 & \text{otherwise.}\end{cases} \tag{2.25}$$

Diagonalisation is simply achieved by Fourier transforms of kernel

$$\mathcal{F}_n(x,p;x',y) = e^{-iyp}\delta_{xx'} \tag{2.26}$$

where

$$p \in \hat{Y}_n \equiv (\hat{h}_{yn}\mathbb{Z}) \cap \left[-\frac{\pi}{h_y}, \frac{\pi}{h_y}\right), \quad \text{and} \quad \hat{h}_y = \frac{\pi}{L_{yn}}. \tag{2.27}$$

Partial Fourier transforms in the p variable block-diagonalize the joint generator, reducing it to the form

$$\left(\mathcal{F}_n\mathcal{L}^{m,n}\mathcal{F}_n^{-1}\right)(x,p;x',p') = \hat{\mathcal{L}}^{m,n}(x,x';p)\delta_{pp'} \tag{2.28}$$

where $\hat{\mathcal{L}}^{m,n}(x,x';p)$ is a one-parameter family of matrices indexed by $x,x' \in X_m$. An explicit expression for the joint generator is given in equation (5.6) below. The kernel itself can also be expressed by means of a Fourier transform as follows:
(2.29)
$$u_{mn}(x,p;x',p';t) \equiv \left(\mathcal{F}_n^{-1}\hat{u}_{mn}(t)\mathcal{F}_n\right)(x,p;x',p') = \frac{1}{h_{xm}h_{yn}}\delta_{\hat{Y}_n}(p-p')\exp\left(t\hat{\mathcal{L}}^{m,n}(p)\right)(x,x'),$$

We are interested in the convergence properties of the sequence

$$\hat{u}_m(x,x';p,t) = \lim_{n\to\infty}\frac{1}{h_{yn}}\hat{u}_{mn}(x,p;x',p';t) \tag{2.30}$$

in the limit as $m \to \infty$. To establish convergence, it is necessary to work in the representation of the Fourier transformed kernel. In fact, while the Fourier transformed kernel converges in a very strong graph-norm and the limit is entire analytic in p and has the same smoothness properties in x,x' as the diffusion kernel, the joint kernel expressed with respect to space coordinates can be quite singular even in simple cases. If for instance the functions $a(x)$ and $b(x)$ in (1.2) are zero, then the joint distribution $U(x,y;x',y';t)$ for the pair (x_t,y_t) is concentrated on the line $y_t = 0$. If a partial Fourier transform with respect to the y variable is taken, the result is constant as a function of the dual variable p. This is a simple example of entire analytic dependency in p which however translates into a rough delta type singularity in the coordinate representation.

Let $g(x,x';z)$ be a complex valued function defined for $x,x' \in X_m \times X_m$ and for $z \in \mathbb{C}$ such that $|z|< K$ where $K > 0$. Consider the uniform norm

$$||g||_{m,K,\infty} = \sup_{\substack{x,x' \in X_m \\ |z|<K}} |g(x,x';z)|. \tag{2.31}$$

The graph norm of order $(m,K,\mathcal{L})$ is defined as follows:

$$||g||_{m,K,\mathcal{L}} = ||g_m||_{m,K,\infty} + ||\mathcal{L}_x^m g||_{m,K,\infty} + ||\mathcal{L}_{x'}^{m*} g||_{m,K,\infty}. \tag{2.32}$$

Theorem 1. *Let's assume the function $b(x)$ is integrable and let*

$$B(x) = \int_0^x b(x')dx' \tag{2.33}$$

be a primitive. If the coefficients $\sigma(x)^2$, $\mu(x)$ and the functions $a(x)$ and $B(x)$ are uniformly continuous, then the sequence of Fourier transformed kernels $\hat{u}_m(t)$ can be extended by analyticity in p to an entire operator valued function and is Cauchy with respect to the above graph-norm for all $K > 0$. If in addition these coefficients are Hölder continuous so that $\sigma^2 \in \mathcal{C}^{k_\sigma,\alpha_\sigma}$, $\mu \in \mathcal{C}^{k_\mu,\alpha_\mu}$, $a \in \mathcal{C}^{k_a,\alpha_a}$, $B \in \mathcal{C}^{k_B,\alpha_B}$ and

$$\gamma \equiv \min\{2, k_\sigma+\alpha_\sigma, k_\mu+\alpha_\mu\ , k_a+\alpha_a, k_B+\alpha_B\} > 0 \tag{2.34}$$

then, for all $K > 0$ there is a constant $c(K)$ such that

$$||\hat{u}_m(t) - \hat{u}_{m'}(t)||_{m,K,\mathcal{L}} \le c(K)h_{xm}^\gamma. \tag{2.35}$$

for all $m' > m \ge m_0$.

Corollary 1. *The limit kernel*

$$u(x,x';z) \equiv \lim_{m\to\infty}\frac{1}{h_{xm}}u_m(x,x';z) \tag{2.36}$$

is an entire analytic function of z which is in the domain of the operators $\mathcal{L}_x$ and $\mathcal{L}_y^*$ and all of their powers. The Fourier transformed kernel

$$u(x,x';y) \equiv \int_{-\infty}^{\infty} e^{-ipy}u(x,x';p)\frac{dp}{2\pi} \tag{2.37}$$

exists in the distribution sense for each fixed pair $x,x' \in [-L_x, L_x]$.

Theorem 2. *Let us assume that the coefficients $\sigma(x)^2$, $\mu(x)$ and the functions $a(x)$ and $B(x)$ are uniformly continuous and fix a $K > 0$. Consider the kernels obtained with a fully explicit Euler scheme, i.e.*

$$\hat{u}_m^{\delta t}(x, x'; z, t) = \frac{1}{h_x} \left(1 + \delta t_m \hat{\mathcal{L}}^m(z)\right)^{\left[\frac{t}{\delta t_m}\right]} (x, x'), \tag{2.38}$$

where $z \in \mathbb{C} : |z| < K$. Assume that δt_m satisfies the Courant condition

$$\min_{x,p \in X_m \times \hat{Y}_n} \Re\left(1 + \delta t_m \hat{\mathcal{L}}^m(x, x; z)\right) > 0 \tag{2.39}$$

for all $|z| < K$. Then, there is a constant $c(K)$ such that

$$||\hat{u}_m(t) - \hat{u}_m^{\delta t}(t)||_{m,K,\mathcal{L}} \leq c(K) h_{xm}^2 \tag{2.40}$$

for all $m' > m \geq m_0$.

This two Theorems are proved in the next two sections. An equivalent statement in a different representation which leads to Ito's Lemma, the Cameron-Marin-Girsanov and the Feynman-Kac formulas is given in Section 5.

3. The Renormalization Group Argument

The proof of convergence in graph-norm is based on a path-integral representation of the probability kernel. More precisely, let us defines a *symbolic path* $\gamma = \{\gamma_0, \gamma_1, \gamma_2, ...\}$ as an infinite sequence of sites in X_m such that $\gamma_j = \gamma_{j-1} \pm 1$ for all $j = 1, ...$ Let Γ_m be the set of all symbolic paths in X_m. Then the propagator admits the following representation:

$$u_m(x, x'; t) = \frac{1}{h_{xm}} \sum_{q=1}^{\infty} \sum_{\gamma \in \Gamma_m : \gamma_0 = x, \gamma_q = x'} \int_0^t ds_1 \int_{s_1}^t ds_2 ... \int_{s_{q-1}}^t ds_q \rho(s_1, s_2, ..., s_{q-1}; \gamma) \tag{3.1}$$

where

$$\rho(s_1, s_2, ..., s_{q-1}; \gamma) = e^{-s_1 \mathcal{L}^m(\gamma_0, \gamma_0)} \prod_{j=1}^{q-1} \mathcal{L}^m(\gamma_{j-1}, \gamma_j) e^{-(s_{j+1} - s_j)\mathcal{L}^m(\gamma_j, \gamma_j)} ds_1 ... ds_q. \tag{3.2}$$

and $s_q = t$.

Let $\gamma \in \Gamma_m$ and consider a path $X_m(\cdot; t_1, ...t_q; \gamma) : \mathbb{R}_+ \to X_m$, left continuous and with right limits, taking the values $\{\gamma_0, \gamma_1, \gamma_2, ...\}$ consecutively with jumps occurring at times $t_j, j = 1, ..q-1$, with $0 \leq t_1 \leq t_2 \leq .. \leq t_{q-1} \leq t$. Let $a(x)$ and $b(x)$ be two uniformly continuous functions in $X = [-L_x, L_x]$ and consider the integral

$$\begin{aligned} I(t_1, ...t_{q-1}; \gamma) = \int_0^t \left[a(X_m(s; t_1, ...t_q; \gamma)) \frac{dX_m(s; t_1, ...t_q; \gamma)}{ds} + b(X_m(s; t_1, ...t_q; \gamma))\right] ds \\ = \sum_{j=0}^{q-1} a(\gamma_j)(\gamma_{j+1} - \gamma_j) + \sum_{j=0}^{q} b(\gamma_j)(t_{j+1} - t_j). \end{aligned} \tag{3.3}$$

We are interested in evaluating the joint density

$$\begin{aligned} u_m(x, y; x', y'; t) = \frac{1}{h_{xm}} \sum_{q=1}^{\infty} \sum_{\gamma \in \Gamma_m : \gamma_0 = x, \gamma_q = x'} \int_0^t ds_1 \int_{s_1}^t ds_2 ... \int_{s_{q-1}}^t ds_q \\ \rho(s_1, s_2, ..., s_{q-1}; \gamma) \delta\big(I(s_1, ...s_{q-1}; \gamma) - (y' - y)\big) \end{aligned} \tag{3.4}$$

We do so by expanding its Fourier transform

(3.5)

$$\hat{u}_m(x,x';p;t) = \frac{1}{h_{xm}} \sum_{q=1}^{\infty} \sum_{\gamma\in\Gamma_m:\gamma_0=x,\gamma_q=x'} \int_0^t ds_1 \int_{s_1}^t ds_2 ... \int_{s_{q-1}}^t ds_q \rho(s_1,s_2,...,s_{q-1};\gamma) e^{ipI(s_1,...s_{q-1};\gamma)}$$

The partial Fourier transform of the joint generator in (2.23) in the limit as $n\to\infty$ is given by

$$\begin{aligned}\hat{\mathcal{L}}^m(x,x';p) &= \lim_{n\to\infty} \hat{\mathcal{L}}^{mn}(x,x';p) \\ &= \lim_{n\to\infty} \sum_{y\in Y_n} \tilde{\mathcal{L}}^{mn}(x,0;x',y;t) e^{-ih_{yn}py} = \left(\frac{\sigma(x)^2}{2h_{xm}^2} + \frac{\mu(x)}{2h_{xm}}\right) e^{-ih_{xm}a(x)p}\delta_{x',x+h_{xm}} \\ &\quad + \left(\frac{\sigma(x)^2}{2h_{xm}^2} - \frac{\mu(x)}{2h_{xm}}\right) e^{ih_{xm}a(x)p}\delta_{x',x-h_{xm}} - \frac{\sigma(x)^2}{h_{xm}^2}\delta_{x'x} - ipb(x)\delta_{x'x} \\ &= \frac{\sigma(x,p,h_{xm})^2}{2}\Delta_x^m(x,x') + \mu(x,p,h_{xm})\nabla_x^m(x,x') + \kappa(x,p,h_{xm})\delta_{xx'}\end{aligned}$$

(3.6)

where

$$\sigma(x,p,h_{xm})^2 = \sigma(x)^2\cos(h_{xm}a(x)p) - i\sin(h_{xm}pa(x))\mu(x) \tag{3.7}$$

$$\mu(x,p,h_{xm}) = \mu(x)\cos(h_{xm}a(x)p) - i\frac{\sin(h_{xm}pa(x))}{h_{xm}}\sigma(x)^2, \tag{3.8}$$

$$\kappa(x,p,h_{xm}) = -i\frac{\sin(h_{xm}pa(x))}{h_{xm}}\mu(x) + \sigma(x)^2(\cos(h_{xm}a(x)p) - 1) - ipb(x). \tag{3.9}$$

Notice that these three functions admit continuations as entire analytic functions in p. Let us denote them with $\sigma(x,z,h_{xm})^2, \mu(x,z,h_{xm}), \kappa(x,z,h_{xm})$. Let $\hat{\mathcal{L}}^m(x,x';z)$ be the analytic continuation of the operator above and consider the Fourier transformed joint propagator defined as follows:

$$\hat{u}_m(x,x';z;t) = \frac{1}{h_{xm}} \sum_{q=1}^{\infty} \sum_{\gamma\in\Gamma_m:\gamma_0=x,\gamma_q=x'} \int_0^t ds_1 \int_{s_1}^t ds_2 ... \int_{s_{q-1}}^t ds_q \rho(s_1,s_2,...,s_{q-1};z;\gamma)$$

(3.10)

where

$$\rho(s_1,s_2,...,s_{q-1};z;\gamma) = \frac{1}{h_{xm}} e^{-s_1\hat{\mathcal{L}}^m(\gamma_0,\gamma_0;z)} \prod_{j=1}^{q-1} \hat{\mathcal{L}}^m(\gamma_{j-1},\gamma_j;z) e^{-(s_{j+1}-s_j)\hat{\mathcal{L}}^m(\gamma_j,\gamma_j;z)} ds_1...ds_q.$$

(3.11)

and $s_q = t$.

If $K>0$, let us introduce the constants

$$\Sigma_0 = \inf_{x\in X_m} \sigma(x,0,h_{xm}), \qquad M(K) = \sup_{\substack{x\in X_m \\ |z|<K}} |\mu(x,z,h_{xm})| \tag{3.12}$$

and

$$\Sigma_1(K) = \sup_{\substack{x\in X_m \\ |z|<K}} \sqrt{|\sigma(x,z,h_{xm})|^2 + |\sigma(x,z,h_{xm}) - \sigma(x,0,h_{xm})|^2 + h_{xm}|\mu(x,z,h_{xm})|}. \tag{3.13}$$

Due to our assumptions, $\Sigma_0 > 0$ and $\Sigma_1(K), M(K) < \infty$.

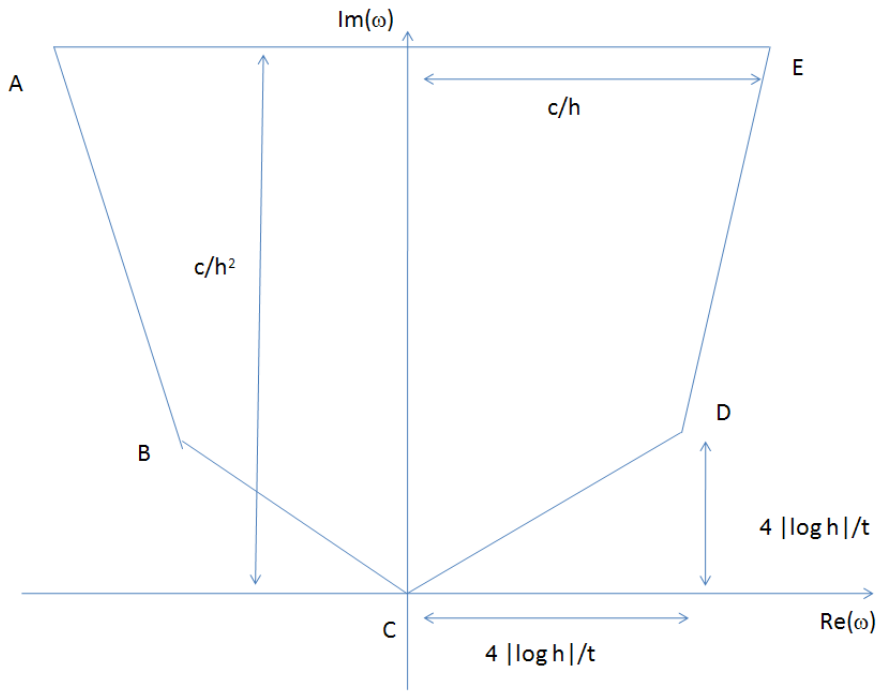

FIGURE 1. Contour of integration for the integral in (3.66). $\mathcal{C}_+$ is the countour joining the point D to the points E, A, B. $\mathcal{C}_-$ is the countour joining the point B to C to D.

A *symbolic path* $\gamma = \{\gamma_0, \gamma_1, \gamma_2, ...\}$ is an infinite sequence of sites in X_m such that $\gamma_j \neq \gamma_{j-1}$ for all $j = 1, ...$ Let Γ_m be the set of all symbolic paths in X_m. The kernel of the diffusion process admits the following representation in terms of a summation over symbolic paths:

$$\hat{u}_m(x, y; z; t) = \sum_{q=1}^{\infty} 2^{-q} \sum_{\substack{\gamma \in \Gamma_m : \gamma_0 = x, \gamma_q = y \\ |\gamma_j - \gamma_{j-1}| = 1 \ \forall j \geq 1}} W_m(\gamma, q, z, t) \tag{3.14}$$

where

(3.15)
$$W_m(\gamma, q, z, t) = \frac{1}{h_{xm}} \int_0^t ds_1 \int_{s_1}^t ds_2 ... \int_{s_{q-1}}^t ds_q e^{(t-s_q)\hat{\mathcal{L}}^m(\gamma_q,\gamma_q;z)} \prod_{j=0}^{q-1} 2\Big(\hat{\mathcal{L}}^m(\gamma_j, \gamma_{j+1}; z) e^{(s_{j+1}-s_j)\hat{\mathcal{L}}^m(\gamma_j,\gamma_j;z)}\Big)$$

with $s_0 = 0$.

Let us introduce the following Green's function:

$$G_m(x, y; z, \omega) = \int_0^{\infty} \hat{u}_m(x, y; z, t) e^{-i\omega t} dt = h_{xm}^{-1} \frac{1}{\hat{\mathcal{L}}^m(z) + i\omega}(x, y). \tag{3.16}$$

The Fourier transformed propagator can be re-obtained from the Green's function by evaluating the following contour integral:

$$\hat{u}_m(x, y; z, t) = \int_{\mathcal{C}_-} \frac{d\omega}{2\pi} G_m(x, y; z, \omega) e^{i\omega t} + \int_{\mathcal{C}_+} \frac{d\omega}{2\pi} G_m(x, y; z, \omega) e^{i\omega t}. \tag{3.17}$$

Here, $\mathcal{C}_+$ is the contour joining the point D to the points E, A, B in Fig. 1, while $\mathcal{C}_-$ is the contour joining the point B to C to D. The intent of the design of the contour of integration is to ensure that each point ω on the upper path $\mathcal{C}_+$ is separated from the spectrum of $\mathcal{L}$.

Lemma 1. *For all $K > 0$, there are an integer $m(K)$ and a constant $c(K)$ such that*

$$\left| \int_{\mathcal{C}_+} \frac{d\omega}{2\pi} G_m(x, y; z, \omega) e^{i\omega t} \right| \leq c(K) h^2. \tag{3.18}$$

for all z such that $|z|< K$.

Proof. The proof is based on the geometric series expansion

$$G_m(z, \omega) = h_m^{-1} \frac{1}{\mathcal{L}^m(z) + i\omega} = h_m^{-1} \sum_{j=0}^{\infty} \frac{1}{\frac{1}{2}\sigma(0, h_{xm})^2 \Delta^m + i\omega} \left[\left(\frac{1}{2}(\sigma(z, h_{xm})^2 - \sigma(0, h_{xm})^2) \Delta^m + \mu(z, h_{xm}) \nabla^m + \kappa(z, h_{xm}) \right) \frac{1}{\frac{1}{2}\sigma(0, h_{xm})^2 \Delta^m + i\omega} \right]^j \tag{3.19}$$

Here $\sigma(z, h_{xm})^2$ and $\mu(z, h_{xm})$ are the multiplication operators by $\sigma(x, z, h_{xm})^2$ and $\mu(x, z, h_{xm})$, respectively. Convergence for $\omega \in \mathcal{C}_+$ can be established by means of a Kato-Rellich type relative bound, see (Kato 1966). More precisely, for any $\alpha > 0$, one can find a $\beta > 0$ such that the operators ∇^m and Δ^m satisfy the following relative bound estimate:

$$||\nabla^m f||_2 \leq \alpha ||\Delta^m f||_2 + \beta ||f||_2. \tag{3.20}$$

for all periodic functions f and all $m \geq m_0$. This bound can be derived by observing that ∇^m and Δ^m can be diagonalized simultaneously by a Fourier transform and by observing that for any $\alpha > 0$, one can find a $\beta > 0$ such that

$$\left| \frac{\sin h_{xm} k}{h_{xm}} \right| \leq \alpha \left| \frac{\cos h_{xm} k - 1}{h_{xm}^2} \right| + \beta \tag{3.21}$$

for all $m \geq m_0$ and all $k \in B_m$.

Under the same conditions, we also have that

$$\left|\left| \left(\frac{1}{2}(\sigma(z, h_{xm})^2 - \sigma(0, h_{xm})^2) \Delta^m + \mu(z, h_{xm}) \nabla^m \right) f \right|\right|_2 \leq \frac{2\alpha}{\Sigma_0^2} \left(M(K) + 2\Sigma_1(K)^2 \right) \left|\left| \frac{1}{2}\sigma(0)^2 \Delta^m f \right|\right|_2 + \beta ||f||_2. \tag{3.22}$$

Hence

$$\left|\left| \mu \nabla^m \frac{1}{\frac{1}{2}\sigma^2 \Delta^m + i\omega} f \right|\right|_2 \leq \left(\frac{1}{2}(\sigma(z, h_{xm})^2 - \sigma(0, h_{xm})^2) \Delta^m + \mu(z, h_{xm}) \nabla^m \right) \left|\left| \frac{1}{2}\sigma^2 \Delta^m \frac{1}{\frac{1}{2}\sigma^2 \Delta^m + i\omega} f \right|\right|_2 + \beta \left|\left| \frac{1}{\frac{1}{2}\sigma^2 \Delta^m + i\omega} f \right|\right|_2 < 1 \tag{3.23}$$

where the last inequality holds if $\omega \in \mathcal{C}_+$, if α is chosen sufficiently small and if m is large enough. In this case, the geometric series expansion in (3.19) converges in L^2 operator norm. The uniform norm of the kernel $|G_m(x, y; \omega)|$ is pointwise bounded from above by h_m^{-2}.

Since the points B and D have imaginary part of height $4 \frac{|\log h_m|}{t}$, the integral over the contour $\mathcal{C}_+$ converges also and is bounded from above by $c h_m^2$ in uniform norm.

□

Lemma 2. *For all $K > 0$, if $q \geq \frac{e^2 t}{2h_m^2}(2\Sigma_1(K)^2 + M(K))$ we have that*

$$W_m(\gamma, q; z, t) \leq \sqrt{\frac{q}{2\pi}} \exp\left(-\frac{\Sigma_0^2 t}{2} - q \right). \tag{3.24}$$

Proof. Let us define the function

$$\phi(t) = \frac{\Sigma_1(K)^2}{2h_m^2}\, e^{-\frac{\Sigma_0^2 t}{2h_m^2}}\, 1(t\geq 0) \tag{3.25}$$

where $1(t\geq 0)$ is the characteristic function of $\mathbb{R}_+$. We have that

$$W_m(\gamma, q; z, t) \leq \phi^{\star q}(t) \tag{3.26}$$

for all z such that $|z|< K$, where $\phi^{\star q}$ is the $q-$th convolution power, i.e. the $q-$fold convolution product of the function ϕ by itself. The Fourier transform of $\phi(t)$ is given by

$$\hat{\phi}(\omega) = \frac{\Sigma_1(K)^2}{2h_m^2}\int_0^\infty e^{-i\omega t - \frac{\Sigma_0^2 t}{2h_m^2}} dt = \frac{\Sigma_1(K)^2}{2i\omega h_m^2 + \Sigma_0^2}. \tag{3.27}$$

The convolution power is given by the following inverse Fourier transform:

$$\phi^{\star q}(t) = \int_{-\infty}^{\infty} \hat{\phi}(\omega)^q e^{i\omega t}\frac{d\omega}{2\pi} = \left(\frac{\Sigma_1(K)}{\Sigma_0}\right)^{2q}\int_{-\infty}^{\infty}\left(1+\frac{2i\omega h_m^2}{\Sigma_0^2}\right)^{-q} e^{i\omega t}\frac{d\omega}{2\pi}. \tag{3.28}$$

Introducing the new variable $z = 1 + \frac{2i\omega h_m^2}{\Sigma_0^2}$, the integral can be recast as follows

$$\phi^{\star q}(t) = \frac{\Sigma_0^{2-2q}\Sigma_1(K)^{2q}}{4\pi i h_m^2}\lim_{R\to\infty}\int_{\mathcal{C}_R} z^{-q}\exp\left(\frac{\Sigma_0^2 t}{2h_m^2}(z-1)\right) dz \tag{3.29}$$

where $\mathcal{C}_R$ is the contour in Fig. 2. Using the residue theorem and noticing that the only pole of the integrand is at $z=0$, we find

$$\phi^{\star q}(t) = \frac{1}{(q-1)!}\left(\frac{\Sigma_1(K)^2 t}{2h_m^2}\right)^q \exp\left(\frac{-\Sigma_0^2 t}{2h_m^2}\right). \tag{3.30}$$

Making use of Stirling's formula $q! \approx \sqrt{2\pi} q^{q+\frac{1}{2}} e^{-q}$, we find

$$\phi^{\star q}(t) \approx \sqrt{\frac{q}{2\pi}}\exp\left(-\frac{\Sigma_0^2 t}{2h_m^2} + q\log\frac{\Sigma_1(K)^2 t}{2h_m^2} + q(1-\log q)\right). \tag{3.31}$$

If $\log q \geq \log\frac{\Sigma_1(K)^2 t}{2h_m^2} + 2$, then we arrive at the bound in (3.24).

□

Let us fix a $K>0$. To prove the theorem, it suffices to consider the case $m' = m+1$ for all values of m above m_0. In fact, given this particular case, the general statement can be derived with an iterative argument. To this end, we introduce a renormalization group transformation based on the notion of decorating path.

Definition 3. (Decorating Paths.) *Let $m\geq m_0$ and let $\gamma = \{y_0, y_1, y_2, ...\}$ be a symbolic sequence in Γ_m. A decorating path around γ is defined as a symbolic sequence $\gamma' = \{y_0, y_1', y_2', ...\}$ with $y_i' \in h_{m+1}\mathbb{Z}$ containing the sequence γ as a subset and such that if $y_j' = y_i$ and $y_k' = y_{i+1}$, then all elements y_n' with $j<n<k$ are such that $|y_n' - y_j'|\leq h_{m+1}$. Let $\mathcal{D}_{m+1}(\gamma)$ be the set of all decorating sequences around γ. The decorated weights are defined as follows:*

$$\tilde{W}_m(\gamma, q; z; t) = \sum_{q'=q}^{\infty}\sum_{\substack{\gamma' \in \mathcal{D}_{m+1}(\gamma)\\ \gamma'_{q'} = \gamma_q}} W_{m+1}(\gamma', q'; z; t). \tag{3.32}$$

Let us notice that these weights are not positive or even real values unless $z=0$, but rather the depend analytically on the variable z. Finally, let us introduce also the following Fourier transform:

$$\hat{W}_m(\gamma, q; z; \omega) = \int_0^\infty W_m(\gamma, q; z; t) e^{i\omega t} dt, \quad \hat{\tilde{W}}_m(\gamma, q; z; \omega) = \int_0^\infty \tilde{W}_m(\gamma, q; z; t) e^{i\omega t} dt. \tag{3.33}$$

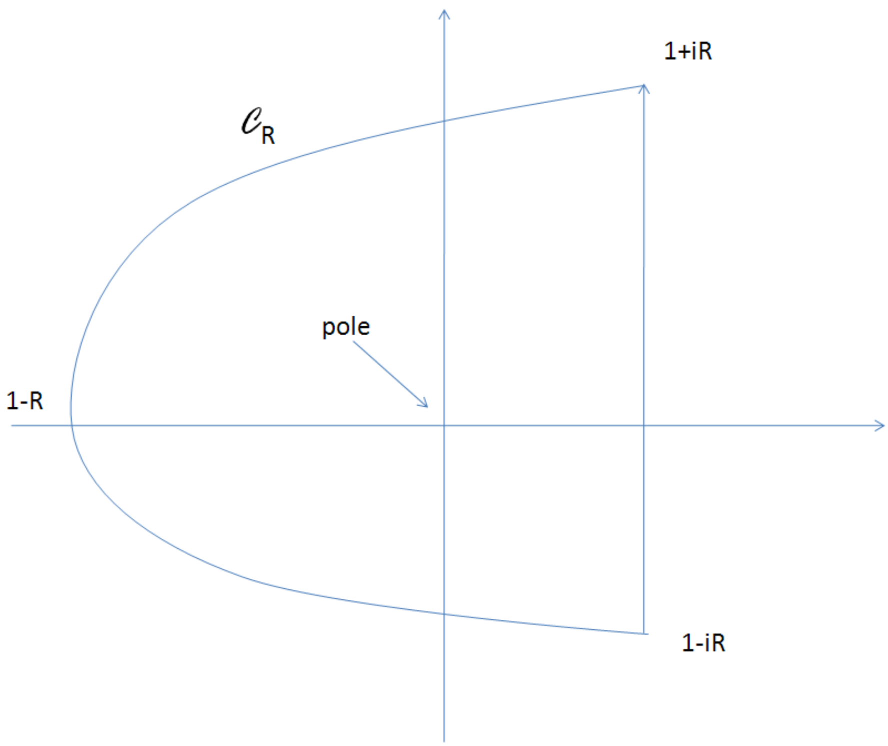

FIGURE 2. Contour of integration $\mathcal{C}_R$ for the integral in (3.29).

Notation 1. *In the following, we set $h = h_{x,m+1}$ so that $h_{xm} = 2h$. We also use the Landau notation $O(h^n)$ to indicate a function $f(h)$ such that $h^{-n}f(h)$ is bounded in a neighborhood of 0.*

Lemma 3. *Let $x, y \in X_m$ and let $\mathcal{C}_-$ be an integration contour as in Fig. 1. Then*

$$\left| \left(\int_{\mathcal{C}_-} 2G_{m+1}(x,y;z;\omega) - G_m(x,y;z;\omega) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right| = O(h^2). \tag{3.34}$$

Proof. We have that

$$2G_{m+1}(x,y;z;\omega) - G_m(x,y;z;\omega) = \frac{1}{h} \sum_{q=1}^{\infty} 2^{-q} \sum_{\substack{\gamma \in \Gamma_m : \gamma_0 = x, \gamma_q = y \\ |\gamma_j - \gamma_{j-1}| = 1 \forall j \geq 1}} \left(2\hat{\hat{W}}_m(\gamma,q;z;\omega) - \hat{W}_m(\gamma,q;z;\omega) \right). \tag{3.35}$$

The number of paths over which the summation is extended is

$$N(\gamma,q;x,y) \equiv \sharp\{\gamma \in \Gamma_m : \gamma_0 = x, \gamma_q = y, |\gamma_j - \gamma_{j-1}| = 1 \forall j \geq 1\} = \binom{q}{\frac{q}{2}+k} \tag{3.36}$$

where $k = \frac{|y-x|}{h_{xm}}$. Applying Stirling's formula we find

$$N_\gamma \lesssim 2^q \sqrt{\frac{2}{\pi q}}. \tag{3.37}$$

Hence

$$\Big|\int_{\mathcal{C}_-}\Big(2G_{m+1}(x,y;z;\omega)-G_m(x,y;z;\omega)\Big)e^{i\omega t}\frac{d\omega}{2\pi}\Big|$$
$$\le \frac{c}{h}\sum_{q=1}^{\infty}\sqrt{\frac{1}{q}}\max_{\substack{\gamma\in\Gamma_m:\gamma_0=x,\gamma_q=y\\|\gamma_j-\gamma_{j-1}|=1\forall j\ge 1}}\Big|\int_{\mathcal{C}_-}\Big(2\hat{\hat{W}}_m(\gamma,q;z,\omega)-\hat{W}_m(\gamma,q;z;\omega)\Big)e^{i\omega t}\frac{d\omega}{2\pi}\Big|. \tag{3.38}$$

for some constant $c\approx\sqrt{\frac{2}{\pi}}>0$. It suffices to extend the summation over q only up to

$$q_{\max}\equiv\frac{e^2\Sigma_1(K)^2t}{2h^2}. \tag{3.39}$$

To resum beyond this threshold, one can use the previous lemma. More precisely, we have that

$$\Big|\int_{\mathcal{C}_-}\Big(2G_{m+1}(x,y;z;\omega)-G_m(x,y;z;\omega)\Big)e^{i\omega t}\frac{d\omega}{2\pi}\Big|$$
$$\le \frac{c\sqrt{q_{\max}}}{h}\max_{\substack{q,\gamma\in\Gamma_m:\gamma_0=x,\gamma_q=y\\|\gamma_j-\gamma_{j-1}|=1\forall j\ge 1}}\Big|\int_{\mathcal{C}_-}\Big(2\hat{\hat{W}}_m(\gamma,q;z,\omega)-\hat{W}_m(\gamma,q;z,\omega)\Big)e^{i\omega t}\frac{d\omega}{2\pi}\Big|. \tag{3.40}$$

Let us introduced the following abbreviated notations:

$$v(x,z)=\sigma(x,z;,h)^2 \tag{3.41}$$
$$m(x,z)=\mu(x,z;,h) \tag{3.42}$$
$$k(x,z)=\kappa(x,z;,h) \tag{3.43}$$

where for the sake of keeping formulas short, we omit to denote the $h=h_{x,m+1}$ dependencies. To evaluate the resummed weight function, let us form the matrix

$$\bar{\mathcal{L}}(x,z)=\begin{pmatrix}-\frac{v(x+h,z)}{h^2}+\kappa\,(x+h,z) & \frac{v(x+h,z)}{2h^2}-\frac{m(x+h,z)}{2h} & 0\\ \frac{v(x,z)}{2h^2}+\frac{m(x,z)}{2h} & -\frac{v(x,z)}{h^2}+\kappa\,(x,z) & \frac{v(x,z)}{2h^2}-\frac{m(x,z)}{2h}\\ 0 & \frac{v(x-h,z)}{2h^2}+\frac{m(x-h,z)}{2h} & -\frac{v(x-h,z)}{h^2}\kappa\,(x-h,z)\end{pmatrix} \tag{3.44}$$

and decompose it as follows:

$$\bar{\mathcal{L}}(x,z)=\frac{1}{h^2}\bar{\mathcal{L}}_0(x,z)+\frac{1}{h}\bar{\mathcal{L}}_1(x,z)+\bar{\mathcal{L}}_2(x,z)+h\bar{\mathcal{L}}_3(x,z). \tag{3.45}$$

where

$$\bar{\mathcal{L}}_0(x,z)=\begin{pmatrix}-v(x,z) & \frac12 v(x,z) & 0\\ \frac12 v(x,z) & -v(x,z) & \frac12 v(x,z)\\ 0 & \frac12 v(x,z) & -v(x,z)\end{pmatrix}, \tag{3.46}$$

$$\bar{\mathcal{L}}_1(x,z)=\begin{pmatrix}-\nabla_x^m v(x,z) & \frac12\nabla_x^m v(x,z)-\frac12 m(x,z) & 0\\ \frac12 m(x,z) & 0 & -\frac12 m(x,z)\\ 0 & -\frac12\nabla_x^m v(x,z)+\frac12 m(x,z) & \nabla_x^m v(x,z)\end{pmatrix}, \tag{3.47}$$

$$\bar{\mathcal{L}}_2(x,z)=\begin{pmatrix}-\frac12\Delta_x^m v(x,z)+k(x,z) & \frac14\Delta_x^m v(x,z)-\frac12\nabla_x^m m(x,z) & 0\\ 0 & k(x,z) & 0\\ 0 & \frac14\Delta_x^m v(x,z)-\frac12\nabla_x^m m(x,z) & -\frac12\Delta_x^m v(x,z)+k(x,z)\end{pmatrix}. \tag{3.48}$$

and

$$\bar{\mathcal{L}}_3(x,z)=\begin{pmatrix}\nabla_x^m k(x,z) & -\frac14\Delta_x^m m(x,z) & 0\\ 0 & 0 & 0\\ 0 & \frac14\Delta_x^m m(x,z) & -\nabla_x^m k(x,z)\end{pmatrix}. \tag{3.49}$$

Recall that all functions and operators above depend also on $h = h_{x,m+1}$.

Let us introduce the sign variable $\tau = \pm 1$, the functions

$$\phi_0(t,x,z,\tau) \equiv 2\bar{\mathcal{L}}_z^m(x, x+2\tau h; z) e^{t\hat{\mathcal{L}}_z^m(x,x;z)} 1(t \geq 0) \tag{3.50}$$

$$\phi_1(t,x,z,\tau) \equiv 2\bar{\mathcal{L}}_z^{m+1}(x+\tau h, x+2\tau h; z) e^{t\bar{\mathcal{L}}(x,z)}(x, x+\tau h) 1(t \geq 0) \tag{3.51}$$

and their Fourier transforms

$$\hat{\phi}_0(\omega,x,z,\tau) = \left(\frac{v(x,z)}{4h^2} + \tau\frac{m(x,z)}{2h}\right)\left(\frac{v(x,z)}{4h^2} - k(x,z) + i\omega\right)^{-1}$$

$$\hat{\phi}_1(\omega,x,z,\tau) = \left(\frac{v(x,z)}{h^2} + \tau\frac{m(x,z)+\nabla_x^m v(x,z)}{h} + \frac{\Delta_x^m v(x,z) + \nabla_x^m m(x,z)}{2} + \frac{\Delta_x^m m(x,z)}{2}\tau h + O(h^2)\right)$$

$$< x|\left(-\bar{\mathcal{L}}(x,z) + i\omega\right)^{-1}|x+\tau h > . \tag{3.52}$$

where

$$|x >= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad |x+\tau h >= \begin{pmatrix} \delta_{\tau,1} \\ 0 \\ \delta_{\tau,-1} \end{pmatrix}. \tag{3.53}$$

We also require the functions

$$\psi_0(t,x,z) \equiv e^{t\mathcal{L}_z^m(x,x;z)} 1(t \geq 0), \qquad \psi_1(t,x) \equiv e^{t\bar{\mathcal{L}}(y;h)}(x,x) 1(t \geq 0) \tag{3.54}$$

and the corresponding Fourier transforms

$$\hat{\psi}_0(\omega,x,z) = \left(\frac{v(x,z)}{4h^2} + i\omega\right)^{-1}, \qquad \hat{\psi}_1(\omega,x,z) =< x|\left(-\bar{\mathcal{L}}(x,z) + i\omega\right)^{-1}|x > . \tag{3.55}$$

If γ is a symbolic sequence, then

$$\hat{W}_m(\gamma,q;z,\omega) = \hat{\psi}_0(\omega,\gamma_q,z)\prod_{j=0}^{q-1}\hat{\phi}_0(\omega;\gamma_j,z,\text{sgn}(\gamma_{j+1}-\gamma_j)) \tag{3.56}$$

$$\hat{\hat{W}}_m(\gamma,q;z;\omega) = \hat{\psi}_1(\omega,\gamma_q,z)\prod_{j=0}^{q-1}\hat{\phi}_1(\omega;\gamma_j,p,\text{sgn}(\gamma_{j+1}-\gamma_j)). \tag{3.57}$$

Let us estimate the difference between the functions $\hat{\phi}_1(\omega,x,z,\tau)$ and $\hat{\phi}_2(\omega,x,z,\tau)$ assuming that ω is in the contour $\mathcal{C}_-$ in Fig. 2. Retaining only terms up to order up to $O(h^3)$, we find

$$\hat{\phi}_0(\omega,x,z,\tau) = 1 + \frac{2m(x,z)\tau h}{v(x,z)} + \frac{4h^2}{v(x,z)}(k(x,z)-i\omega) + \frac{8m(x,z)\tau h^3}{v(x,z)^2}(k(x,z)-i\omega) + \frac{16h^4}{v(x,z)^2}(k(x,z)-i\omega)^2 + O(h^5). \tag{3.58}$$

A lengthy but straightforward calculation which is best carried out using a symbolic manipulation program, gives

$$\hat{\phi}_1(\omega,x,z,\tau) = 1 + \frac{2m(x,z)\tau h}{v(x,z)} + \frac{4h^2}{v(x,z)}(k(x,z)-i\omega) - \left[8m(x,z) - \nabla_x^m v(x,z)\right]\frac{i\omega\tau h^3}{v(x,z)^2} + R(x,z)\cdot h^3\tau + h^4 P_0(x,z) + i\omega h^4 P_1(x,z) - \frac{14\omega^2 h^4}{v(x,z)^2} + O(h^5) \tag{3.59}$$

where

$$R = \frac{1}{2v^3}\Big[\big(\Delta_x^m m + 2\nabla_x^m k\big)v^2 + \big(16m - 2\nabla_x^m v\big)kv$$
$$- 4m^3 + 2\nabla_x^m v m^2 - 2v\nabla_x^m v \nabla_x^m m - \Big(2m\nabla_x^m m + v\Delta_x^m v - 2\big(\nabla_x^m v\big)^2\Big)m\Big].$$
$$P_0 = \frac{1}{v^3}\bigg[\Big(\big(2m - 2\nabla_x^m v\big)\nabla_x^m k + 14k^2 + (-4\nabla_x^m m - \Delta_x^m v)k\Big)v + \Big(- 4m^2 + 2m\nabla_x^m v + 2(\nabla_x v)^2\Big)k\bigg]$$
$$P_1 = \frac{1}{v^3}\Big[28k - 4m^2 + 2m\nabla_x^m v - 4v\nabla_x m - v\Delta_x^m v + 2\big(\nabla_x^m v\big)^2\Big].$$
(3.60)

For simplicity, we are not denoting here the dependency of all functions on (x, z). We have that

$$\sum_{j=0}^{q-1}\Big(\log\hat\phi_0(\omega;\gamma_j,z,\mathrm{sgn}(\gamma_{j+1}-\gamma_j)) - \log\hat\phi_1(\omega;\gamma_j,z,\mathrm{sgn}(\gamma_{j+1}-\gamma_j))\Big)$$
$$= \sum_{j=0}^{q-1}\Big(\frac{i\omega\nabla_x^m v(\gamma_j,z)}{v(\gamma_j,z)^2} + R(\gamma_j,z)\Big)h^3\mathrm{sgn}(\gamma_{j+1}-\gamma_j) + \big(||P_0||_{K,\infty} + |\omega|||P_1||_{K,\infty} + 2|\omega|^2||v^{-2}||_{K,\infty}\big)O(h^4q)$$
$$= i\omega h^2\Big(\frac{1}{v(\gamma_0,z)} - \frac{1}{v(\gamma_q,z)}\Big) + h^2\big(G(\gamma_q,z) - G(\gamma_0,z)\big) + \big(||P_0||_{K,\infty} + |\omega|||P_1||_{K,\infty} + 2|\omega|^2||v^{-2}||_{K,\infty}\big)O(h^4q)$$
(3.61)

where $G(x,z)$ is a primitive of $R(x,z)$, i.e.

$$G(x,z) = \int_{-L_x}^{x} R(x',z)dx'. \tag{3.62}$$

We conclude that there is a constant $c(K) > 0$ such that

(3.63)
$$\Big|\int_{\mathcal{C}_-}\Big(\prod_{j=0}^{q-1}\hat\phi_0(\omega;\gamma_j,z,\mathrm{sgn}(\gamma_{j+1}-\gamma_j)) - \prod_{j=0}^{q-1}\hat\phi_1(\omega;\gamma_j,z,\mathrm{sgn}(\gamma_{j+1}-\gamma_j))\Big)e^{i\omega t}\frac{d\omega}{2\pi}\Big| \le c(K)h^\gamma.$$

for all $q \le q_{\max}$. Here we use the decay of $e^{i\omega t}$ in the upper half of the complex ω plane to offset the ω dependencies in the integrand. Similar calculations lead to the following expansions:

(3.64)
$$\hat\psi_0(\omega,x,z) = \frac{4h^2}{v(x,z)} + O(\omega h^4), \qquad \hat\psi_1(\omega,x,z) = \frac{2h^2}{v(x,z)} + O(\omega h^4) = \frac{1}{2}\hat\psi_0(\omega,x,z) + O(\omega h^4).$$

Since $q < ch^{-2}$ and $\omega \le |\log h|$, we find

$$\Big|\int_{\mathcal{C}_-}\Big(2G_{m+1}(x,y;z,\omega) - G_m(x,y;z,\omega)\Big)e^{i\omega t}\frac{d\omega}{2\pi}\Big| \le c(K)\frac{\sqrt{q_{\max}}}{h}h^{2+\gamma} \le ch^\gamma. \tag{3.65}$$

By differentiating with respect to time in equation (3.66), we find that

$$\frac{\partial}{\partial t}u_m(x,y;t) = \int_{\mathcal{C}_-} i\omega G_m(x,y;z;\omega)e^{i\omega t}\frac{d\omega}{2\pi} + \int_{\mathcal{C}_+} i\omega G_m(x,y;z;\omega)e^{i\omega t}\frac{d\omega}{2\pi}. \tag{3.66}$$

All the derivations above carry through. We conclude that for some constant $c_1(K) > 0$ and all $|z| < K$ we have that

$$\Big|\int_{\mathcal{C}_+}\frac{d\omega}{2\pi} i\omega G_m(x,y;z;\omega)e^{i\omega t}\Big| \le c_1(K)h^\gamma \tag{3.67}$$

and also

$$\left| \int_{\mathcal{C}_-} i\omega \Big(2G_{m+1}(x,y;z;\omega) - G_m(x,y;z;\omega) \Big) e^{i\omega t} \frac{d\omega}{2\pi} \right| \leq c \frac{\sqrt{q_{\max}}}{h} h^4 \leq c_1(K) h^\gamma. \tag{3.68}$$

Hence, the first time derivatives of the kernel satisfy a similar Cauchy convergence condition as the kernel itself.

Let us notice that, if m is large enough, then the operator $\hat{\mathcal{L}}^m(z)$ is relatively bounded with respect to $\mathcal{L}^m$ in the following sense:

$$||\mathcal{L}f|| \leq \alpha(K)||\hat{\mathcal{L}}(z)f|| + \beta(K)||f|| \tag{3.69}$$

for some $\alpha(K), \beta(K) > 0$ and all $|z| < K$. By inspection of the terms in $\hat{\mathcal{L}}(z)$ one sees that such bound holds if h_{xm} is small enough, i.e. for m larger than a threshold depending on K. A similar relative bound holds for the adjunct operator $\mathcal{L}^*$ with respect to $\hat{\mathcal{L}}(z)^*$. This concludes the proof of Theorem 1.

□

4. Explicit Euler Scheme

In this section we prove Theorem 2. A path-wise expansion for the time-discretization of the Fouier transformed kernel has the form

$$\hat{u}_m^{\delta t}(x,y;z,t) = \frac{1}{h_{xm}} \sum_{q=1}^{\infty} \sum_{\gamma \in \Gamma_m : \gamma_0 = x, \gamma_q = y} \sum_{k_1=1}^{N} \sum_{k_2=k_1+1}^{N} .. \sum_{k_q=k_{q-1}+1}^{N}$$

$$\Big(1 + \delta t \hat{\mathcal{L}}_m(\gamma_0,\gamma_0;z)\Big)^{k_1-1} (\delta t)^q \prod_{j=1}^{q} \hat{\mathcal{L}}_m(\gamma_{j-1},\gamma_j;z) \Big(1 + \delta t \hat{\mathcal{L}}_m(\gamma_j,\gamma_j;z)\Big)^{k_{j+1}-k_j-1} \tag{4.1}$$

where $t_{q+1} = t$ and $k_{q+1} = N$. In this case, the propagator can be expressed through a Fourier integral as follows:

$$\hat{u}_m^{\delta t}(x,y;z,t) = \int_{-\frac{\pi}{\delta t}}^{\frac{\pi}{\delta t}} G_m^{\delta t}(x,y;z,\omega) e^{i\omega t} \frac{d\omega}{2\pi} \tag{4.2}$$

where

$$G_m^{\delta t}(x,y;z,\omega) = \delta t \sum_{j=0}^{\infty} \hat{u}_m^{\delta t}(x,y;z,j\delta t) e^{-i\omega j \delta t}. \tag{4.3}$$

The Fourier transformed propagator can also be represented as the limit

$$\hat{u}_m^{\delta t}(x,y;z,t) = \lim_{H\to\infty} \int_{\mathcal{C}_H} G_m^{\delta t}(x,y;z,\omega) e^{i\omega t} \frac{d\omega}{2\pi} \tag{4.4}$$

where $\mathcal{C}_H$ is the contour in Fig. 3. This is due to the fact that the integral along the segments BC and DA are the negative of each other, while the integral over CD tends to zero exponentially fast as $\Im(\omega) \to \infty$, where $\Im(\omega)$ is the imaginary part of ω. Using Cauchy's theorem, the contour in Fig. 3 can be deformed into the contour in Fig. 1. To estimate the discrepancy between the time-discretized kernel and the continuous time one, one can thus compare the Green's function along such contour. Again, the only arc that requires detailed attention is the arc BCD, as the integral over rest of the contour of integration can be bounded from above as in the previous section.

Let $h = h_{xm}$ and let us introduce the two functions

$$\phi_0(t,x,z,\tau) \equiv 2\hat{\mathcal{L}}_m(x,x+\tau h;z) e^{t\hat{\mathcal{L}}_m(x,x;z)} 1(t \geq 0), \tag{4.5}$$

$$\phi_{\delta t}(j,x,z,\tau) \equiv 2\hat{\mathcal{L}}_m(x,x+\tau h;z) \big(1 + \delta t \hat{\mathcal{L}}_m(x,x;z)\big)^{j-1}. \tag{4.6}$$

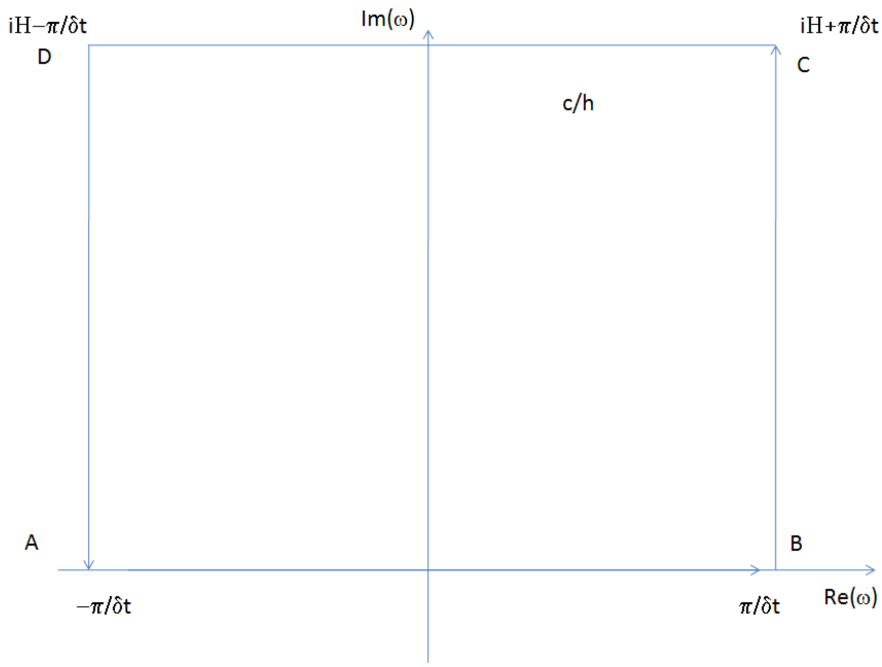

FIGURE 3. Contour of integration for the integral in (4.4).

and the corresponding Fourier transforms

(4.7)
$$\hat{\phi}_0(\omega,x,z,\tau) = \int_0^\infty \phi_0(t,x,z,\tau)e^{-i\omega t}\frac{d\omega}{2\pi} = \left(\frac{v(x,z,h)}{h^2} + \tau\frac{m(x,z,h)}{h}\right)\left(\frac{v(x,z,h)}{h^2} + i\omega\right)^{-1}$$

(4.8)
$$\hat{\phi}_{\delta t}(\omega,x,z,\tau) = \sum_{j=0}^{\frac{t}{\delta t}} \phi_{\delta t}(j,x,z,\tau)e^{-i\omega j\delta t} = \left(\frac{v(x,z,h)}{h^2} + \tau\frac{m(x,z,h)}{h}\right)\left(e^{i\omega\delta t} - 1 + \delta t\frac{v(x,z,h)}{h^2}\right)^{-1}.$$

We have that

$$\hat{\phi}_{\delta t}(\omega,x,z,\tau) = \left(\frac{v(x,z,h)}{h^2} + \tau\frac{m(x,z,h)}{h}\right)\left(i\omega + \frac{v(x,z,h)}{h^2} - \frac{\omega^2}{2}\delta t + O(\delta t^2)\right)^{-1}$$
$$= \hat{\phi}_0(\omega,x,z,\tau) + \frac{\omega^2}{2v(x,z,h)}h^2\delta t + O(h^2\delta t^2). = \hat{\phi}_0(\omega,x,z,\tau) + O(h^4), \tag{4.9}$$

where the last step uses the fact that $\delta t = O(h^2)$.

Let us also introduce the functions

$$\psi_0(t,x,z,\tau) \equiv e^{t\hat{\mathcal{L}}_m(x,x;z)}1(t\geq 0), \qquad \psi_{\delta t}(j,x,\tau) \equiv \sum_{k=1}^{j}\left(1 + \delta t\hat{\mathcal{L}}_m(x,x;z)\right)^{j-1}. \tag{4.10}$$

and the corresponding Fourier transforms

(4.11)
$$\hat{\psi}_0(\omega,x,z,\tau) = \left(\frac{v(x,z,h)}{h^2} + i\omega\right)^{-1}, \qquad \hat{\psi}_{\delta t}(\omega,x,\tau) = \left(e^{i\omega\delta t} - 1 + \delta t\frac{v(x,z,h)}{h^2}\right)^{-1}.$$

Again we find that

$$\hat{\psi}_0(\omega,x,z,\tau) = \hat{\psi}_{\delta t}(\omega,x,z,\tau) + O(h^4). \tag{4.12}$$

If γ is a symbolic sequence, then let us set

$$\hat{W}_m(\gamma, q; z, \omega) = \hat{\psi}_0(\omega, \gamma_q) \prod_{j=0}^{q-1} \hat{\phi}_0(\omega; \gamma_j, z, \text{sgn}(\gamma_{j+1} - \gamma_j)) \tag{4.13}$$

$$\hat{W}_m^{\delta t}(\gamma, q; z, \omega) = \hat{\psi}_{\delta t}(\omega, \gamma_q) \prod_{j=0}^{q-1} \hat{\phi}_{\delta t}(\omega; \gamma_j, z, \text{sgn}(\gamma_{j+1} - \gamma_j)). \tag{4.14}$$

We have that

(4.15)

$$G_m^{\delta t}(x, y; z, \omega) - G_m(x, y; z, \omega) = \frac{1}{h} \sum_{q=1}^{\infty} 2^{-q} \sum_{\substack{\gamma \in \Gamma_m : \gamma_0 = x, \gamma_q = y \\ |\gamma_j - \gamma_{j-1}| = 1 \forall j \geq 1}} \left(\hat{W}_m^{\delta t}(\gamma, q; z, \omega) - \hat{W}_m(\gamma, q; z, \omega) \right).$$

The integration over the contour in Fig. 1 can again be split into an integration over the countour $\mathcal{C}_-$ and an integration over $\mathcal{C}_+$. The integral over $\mathcal{C}_+$ can be bounded from above thanks to Lemma 1. Furthermore, we have that

$$\begin{aligned}
&\left| \int_{\mathcal{C}_-} \left(G_m^{\delta t}(x, y; z, \omega) - G_m(x, y; z, \omega) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right| \\
&\leq ch^{-1} \sqrt{q_{\max}} \max_{\substack{q, \gamma \in \Gamma_m : \gamma_0 = x, \gamma_q = y \\ |\gamma_j - \gamma_{j-1}| = 1 \forall j \geq 1}} \left| \int_{\mathcal{C}_-} \left(\hat{W}_m^{\delta t}(\gamma, q; z, \omega) - \hat{W}_m(\gamma, q; z, \omega) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right|. \\
&\leq ch^2
\end{aligned} \tag{4.16}$$

To bound the time derivative, we have to consider

$$\left| \int_{\mathcal{C}_-} \left(\frac{e^{i\omega\delta t} - 1}{\delta t} G_m^{\delta t}(x, y; z, \omega) - i\omega G_m(x, y; z, \omega) \right) e^{i\omega t} \frac{d\omega}{2\pi} \right| \tag{4.17}$$

But, since $\delta t = O(h^2)$, also this difference is $O(h^2)$.

5. Convergence in the Ito Representation

The convergence Theorem 1 admits a second formulation. Let us introduce the function

$$\phi_m(x) = h_{xm} \sum_{x' \in X_m, x' \leq x} a(x'). \tag{5.1}$$

We have that

$$\begin{aligned}
e^{ip\phi_m(x)} \mathcal{L}(x, x') e^{-ip\phi_m(x')} &= \left(\frac{\sigma(x)^2}{2h_{xm}^2} + \frac{\mu(x)}{2h_{xm}} \right) e^{-ih_{xm} a(x+h_{xm})p} \delta_{x', x+h_{xm}} \\
&+ \left(\frac{\sigma(x)^2}{2h_{xm}^2} - \frac{\mu(x)}{2h_{xm}} \right) e^{ih_{xm} a(x)p} \delta_{x', x-h_{xm}} - \frac{\sigma(x)^2}{h_{xm}^2} \delta_{x'x} \\
&= \frac{1}{2} \sigma(x, p, h_{xm})^2 \Delta^m(x, x') + \mu(x, p, h_{xm}) \nabla^m(x, x') \\
&+ \left[-i \frac{\sin(h_{xm} p a(x))}{h_{xm}} \mu(x) + \frac{1}{2} (\cos(h_{xm} a(x) p) - 1) \right] \delta_{xx'} \\
&+ \left(\frac{\sigma(x)^2}{2} + \frac{h_{xm} \mu(x)}{2} \right) \left(h_{xm}^{-1} \nabla_x^+ e^{-ih_{xm} a(x) p} \right) \left(\delta_{xx'} + h_{xm} \nabla_x^+(x, x') \right)
\end{aligned} \tag{5.2}$$

where $\sigma(x,p,h_{xm})^2$ and $\mu(x,p,h_{xm})$ are defined in (3.7) and (3.8) and

$$\nabla_x^+ f(x) = \frac{f(x+h_{xm}) - f(x)}{h_{xm}}. \tag{5.3}$$

Hence, we have that

$$\tilde{\mathcal{L}}^m(x,x';p) = e^{ip\phi_m(x)} L^m(x,x';p) e^{-ip\phi_m(x')} \tag{5.4}$$

where

$$L^m(x,x';p) = \mathcal{L}(x,x') + \zeta_m(x,p,h_{xm})\delta_{xx'} + h_{xm} r_m(x,x';p). \tag{5.5}$$

Here

$$\zeta_m(x,p,h_{xm}) = ipb(x) + \frac{\sigma(x)^2}{2}\left(h_{xm}^{-1}\nabla_x^+ e^{-ih_{xm}a(x)p}\right),$$
$$r_m(x,x';p) = \left(h_{xm}^{-1}\nabla_x^+ e^{-ih_{xm}a(x)p}\right)\left[\left(\frac{\sigma(x)^2}{2} + \frac{h_{xm}\mu(x)}{2}\right)\nabla_x^+(x,x') + \frac{\mu(x)}{2}\delta_{xx'}\right] e^{ip(\phi_m(x') - ip\phi_m(x))}. \tag{5.6}$$

and

$$\nabla_x^{m+} f(x) = \frac{f(x+h_{xm}) - f(x)}{h_{xm}}. \tag{5.7}$$

The operator $\tilde{L}^m(x,x';p)$ is equivalent to $\tilde{\mathcal{L}}^m(x,x';p)$ up to a non-singular linear transformation. We say that $\tilde{L}^m(x,x';p)$ is the *Fourier transformed generator in the Ito representation.* Also notice that the weak limit of this operator as $m \to \infty$ is equal to

$$\lim_{m\to\infty} L^m(x,x';p) = \frac{\sigma(x)^2}{2}\frac{\partial^2}{\partial x^2} + \mu(x)\frac{\partial}{\partial x} + ipb(x) - \frac{ip\sigma(x)^2}{2}\frac{\partial a(x)}{\partial x}. \tag{5.8}$$

The Fourier transformed kernel is given by

$$\hat{u}_m(x,x';p) = e^{ip(\phi_m(x) - \phi_m(x'))}\hat{U}_m(x,x';p) \tag{5.9}$$

where

$$\hat{U}_m(x,x';p) = h_{xm}^{-1} e^{tL^m}(x,x'). \tag{5.10}$$

The joint kernel is thus given by the following formula whose continuum analog was found in (Girsanov 1960), (Cameron and Martin 1949), (Feynman 1948) and (Ito 1949):

$$u(x,I;x',I';t) = \int_{-\infty}^{\infty}\frac{dp}{2\pi} e^{ip[I'-I-\phi_m(x')+\phi_m(x)]}\hat{U}_m(x,x';p). \tag{5.11}$$

Theorem 3. *Under the same assumptions of Theorem 1, for all $K > 0$ there is a constant $c(K)$ such that*

$$||\hat{U}_m^{\phi}(t) - \hat{U}_{m'}^{\phi}(t)||_{m,K,\hat{\mathcal{L}}} \leq c(K) h_{xm}^{\gamma}. \tag{5.12}$$

for all $m' > m \geq m_0$. A similar bound also holds for the kernels obtained with a fully explicit Euler scheme, i.e.

$$\hat{U}_m^{\delta t}(x,x';p,t) = \frac{1}{h_x}\left(1 + \delta t_m L^m(p)\right)^{\left[\frac{t}{\delta t_m}\right]}(x,x'), \tag{5.13}$$

where δt_m is so small that

$$\min_{x,p\in X_m\times\hat{Y}_n} 1 + \delta t_m \hat{\mathcal{L}}^m(x,x;p) > 0. \tag{5.14}$$

In this case, there is a constant $c(K)$ *such that*

$$||U_m(t) - U_m^{\delta t}(t)||_{m,K,\hat{\mathcal{L}}} \leq c(K) h_{xm}^2 \tag{5.15}$$

for all $m' > m \geq m_0$.

It is possible to retrace the argument in the previous section except for replacing

$$\kappa(x, p, h_{xm}) \to \zeta_m(x, p, h_{xm}) + h_{xm} r_m(x, p, h_{xm}) \tag{5.16}$$

where $\kappa(x, p, h_{xm})$ is the function in (3.9). All arguments go through unaltered under the same conditions. As a consequence we conclude that the kernels

$$\frac{1}{h_{xm}} \exp\Big(t\big(\mathcal{L}^m + \zeta_m(z) + h_{xm} r_m(z)\big)\Big)(x, x') \tag{5.17}$$

converge in graph norm, uniformly on discs $z \in \mathbb{C} : |z| < K$, for any $K > 0$

6. Other Abelian Processes

In this Section, we give to examples of Abelian processes which emerge from applications and are not stochastic integrals.

6.1. **The Sup Process.** Consider the sup process

$$y_t = \sup_{\in [0,t]} x_s. \tag{6.1}$$

the sup is always attained as an element of the underlying space X_m, it is natural in this case to restrict the attention to the case where $n = m$ and $Y_m = X_m$. The joint generator is given by

$$\tilde{\mathcal{L}}^m(x, y; x', y') = \mathcal{L}^m(x; x') \mathcal{A}^m(x, y; x', y') \tag{6.2}$$

where $x, y, x', y' \in X_m$ and we set

$$\mathcal{A}^m(x, y; x', y') = \begin{cases} \delta_{yy'} & \text{if} \quad x' < y \\ \delta_{x'y'} & \text{if} \quad x' \geq y. \end{cases} \tag{6.3}$$

Consider the matrix

$$\mathcal{V}_m(x, y; x', y') = \delta(x - x') 1(y' \geq y) \tag{6.4}$$

and its inverse

$$\mathcal{V}_m^{-1}(x, y; x', y') = \delta(x - x')\big(\delta(y' - y) - \delta(y' - y - h_{xm})\big). \tag{6.5}$$

Consider the one parameter family of operators $\hat{\mathcal{L}}^m(y)$ such that

$$\big(\mathcal{V}_m^{-1} \tilde{\mathcal{L}}^m \mathcal{V}_m\big)(x, y; x', y') = \hat{\mathcal{L}}^m(x, x'; y) \delta_{yy'}. \tag{6.6}$$

We have that

$$\hat{\mathcal{L}}^m(x, x'; y) = \mathcal{L}^m(x; x') 1(x' \leq y). \tag{6.7}$$

Hence, $\hat{\mathcal{L}}^m(x, x'; y)$ is the Markov generator of the underlying process with absorption in the interval $[y, L_x]$.

Let us notice that the kernel can be obtained as follows:

$$u_m(t) = \mathcal{V}_m \tilde{u}_m(t) \mathcal{V}_m^{-1} \quad \text{where} \quad \tilde{u}_m(x, y; x', y'; t) = \delta(y - y') \exp\Big(t \tilde{\mathcal{L}}^m(y)\Big)(x, x'). \tag{6.8}$$

A more explicit way of expressing the joint kernel is

$$u_m(x, y; x', y'; t) = \delta(y - y') \hat{u}_m((x, x'; y, t) + 1(y' > y)\Big(\hat{u}_m(x, x'; y', t) - \hat{u}_m(x, x'; y' - h_{xm}, t)\Big) \tag{6.9}$$

Convergence in this case can be established along the same lines as done for stochastic integrals. The situation is simpler in that only the consideration of the kernel itself, i.e. the $p = z = 0$ case with the notations in Sections 2 and 3, is needed. The additional complication is that we need to consider absorbing boundary conditions. This implies a few marginal changes to the derivation above, as when a path arrives to an absorbing lattice point, it stays constant thereafter. Since this applies both to paths and decorating paths in a finer lattice, upon arriving to an absorption point the dynamics is trivial in either case and the final bounds given still hold.

6.2. **Discrete Time Processes.** This section is based on work in collaboration with Manlio Trovato (Albanese and Trovato 2005) and Paul Jones, see (Albanese and Jones 2007*b*).

An important class of path-dependent options requires computing the joint distribution of the underlying lattice process and of a discrete sum of the following form:

$$y_t = \sum_{i=1}^{N} \psi(x_{t_{i-1}}, x_{t_i}; t_i) \tag{6.10}$$

where N is an integer, $t_i = i\Delta T$ and $T = N\Delta T$. Consider the elementary propagator

$$U_m(x_1, x_2) = \frac{1}{h_{xm}} e^{(\Delta T)\mathcal{L}}(x_1, x_2). \tag{6.11}$$

To find the joint transition probability, one can again discretize the variable y_t in the lattice $Y_n = h_{yn}\mathbb{Z} \cap [-L_y, L_y]$. As opposed to lifting the generator as done above for the other cases, here we lift the elementary propagator itself and form the joint propagator

$$\tilde{U}_{mn}(x_1, y_1; x_2, y_2) = U_m(x_1, x_2)\delta\big(y_1 - y_2 + [\psi(x_1, x_2)h_{yn}^{-1}]\big). \tag{6.12}$$

This lifted operator can be block-diagonalized by means of a partial Fourier transform. Consider the Fourier transform operator $\hat{U}_m(p)$ of matrix elements

$$\hat{U}_m(x_1, x_2; p) = \lim_{n\to\infty} U_{mn}(x_1, y_1; x_2, y_2) e^{-ip(y_2 - y_1)} = U_m(x_1, x_2) e^{-ip\psi(x_1, x_2)}. \tag{6.13}$$

Then we have that

$$\lim_{n\to\infty} \big(\tilde{U}_{mn}^N\big)(x_1, y_1; x_2, y_2) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(y_2 - y_1)} \big(\hat{U}_m^N\big)(x_1, x_2; p) \tag{6.14}$$

Convergence in the graph-norm in this case descends directly from the convergence of the one-period kernel $U_m(x_1, x_2)$.

7. Conclusions

We obtained bounds on convergence rates for explicit discretization schemes to the Fourier transform of joint kernels of one-dimensional diffusion equations with continuous coefficients and a stochastic integral. We consider both semi-discrete triangulations with continuous time and explicit Euler schemes with time step small enough for the method to be stable. The proof is constructive and based on a new technique of path conditioning for Markov chains and a renormalization group argument. Convergence rates depend on the degree of smoothness and Hölder differentiability of the coefficients. The method also applies to a more general class of path dependent processes we call Abelian. Examples of Abelian processes beside stochastic integrals are the sup process and discrete time summations.

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