Effectiveness of CPPI Strategies under Discrete–Time Trading

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Motivation I

- Portfolio insurance strategies:
  - guarantee a minimum level of wealth at a specified time horizon
  - and participate in the potential gains of a reference portfolio
- Most prominent examples of dynamic versions
  - constant proportion portfolio insurance (CPPI)
  - option–based portfolio insurance (OBPI) (with synthetic puts)

Motivation II

- Optimality of an investment strategy depends on

  risk profile of the investor

⇒ solve for the strategy which maximizes the expected utility

- Portfolio insurers can be modelled by

  - utility maximizers with
  - additional constraint that the value of the strategy is above a specified wealth level

⇒ Mostly, solution is given by the unconstrained problem including a put option (in spirit of OPBI method)

Motivation III

- Introduction of various sources of market incompleteness
  - stochastic volatility
  - trading restrictions

⇒ Determination of an optimal investment rule under minimum wealth constraints quite difficult if not impossible

- Another problem is model risk
  - inconsistency between true and assumed model

  ⇒ strategies based on optimality criterion w.r.t. one particular model, fail to be optimal if true asset price dynamic deviate

Motivation IV

- Alternative to maximization approach
  - analysis of robustness properties of a stylized strategy

  ⇒ we consider the CPPI rule as given

- CPPI is very popular among practitioners because of its
  - simplicity and
  - possibility to customize it to the preferences of an investor

- CPPI provides a value above a floor level **unless** price dynamic of the risky asset permits jumps (gap risk)

Motivation V

- Liquidity constraints and price jumps can be modeled in a setup where

  - the price dynamic of the risky asset is described by a continuous–time stochastic process
  - but trading is restricted to discrete time

⇒ Benchmark case with the advantage that

  - risk measures can be given in closed form (gap risk is easily priced)
  - we can discuss criteria which ensure that the gap risk does not increase to a level which contradicts the original intention of portfolio insurance

Outline

- Introduction
  - Model setup
  - Structure and properties of continuous–time CPPI (Review)
- Discrete–time version of CPPI
  - Conditions which define the discrete–time version
  - Risk measures of discrete–time CPPI
  - Introduction of transaction costs and their effects
- Illustration of results

Model Setup

- Two investment possibilities
  - a risky asset S
  - and a riskless bond B which grows with constant interest rate r
- Assumption

$$\begin{aligned} dB_t &= B_t r\, dt, \quad B_0 = b \\ d\, S_t &= S_t \left(\mu\, dt + \sigma\, dW_t\right), \quad S_0 = s \end{aligned}$$

$W = (W_t)_{0 \leq t \leq T}$ denotes a standard Brownian motion with respect to the *real world* measure P. μ and σ are constants ($\mu > r \geq 0$ and $\sigma > 0$)

CPPI principle I

Continuous–time investment strategy or saving plan for the interval $[0, T]$

- α_t: fraction of the portfolio value at time t which is invested in the risky asset S
- $V = (V_t)_{0 \leq t \leq T}$: portfolio value process which is associated with the strategy α, i.e.

$$dV_t(\alpha) = V_t \left(\alpha_t \frac{dS_t}{S_t} + (1 - \alpha_t) \frac{dB_t}{B_t} \right), \text{ where } V_0 = x.$$

CPPI principle II

- CPPI principle

$$\alpha_t := \frac{mC_t}{V_t}$$

where

- $C_t = V_t - \underbrace{F_t}_{\text{Floor}}$ denotes the cushion
- $F_t = \exp\{-r(T-t)\underbrace{G}_{\text{guarantee}}\}$ denotes the floor
- m denotes the multiplier

Properties of continuous–time CPPI I

Lognormal asset price dynamic implies

- Cushion process $(C_t)_{0 \le t \le T}$ of a simple CPPI is lognormal, i.e.

$$dC_t = C_t \left((r + m(\mu - r)) \, dt + \sigma m \, dW_t \right)$$

- t–value of a simple CPPI with parameter m and G is

$$V_t = \begin{array}{c} Ge^{-r(T-t)} \\ + \frac{V_0 - Ge^{-rT}}{S_0^m} \exp\left\{ \left(r - m\left(r - \frac{1}{2}\sigma^2 \right) - m^2 \frac{\sigma^2}{2} \right) t \right\} S_t^m \end{array}$$

Properties of continuous–time CPPI II

- t–value of the strategy consists of
  - the present value of the guarantee G, i.e. the floor at t
  - and a non–negative part which is proportional to $\left(\frac{S_t}{S_0}\right)^m$

$\Rightarrow$ Value process of a simple CPPI strategy is path independent

- The payoff above the guarantee is
  - linear for $m = 1$
  - convex for $m \geq 2$
- Portfolio protection is efficient with probability one, i.e. the terminal value of the strategy is higher than the guarantee

Properties of continuous–time CPPI III

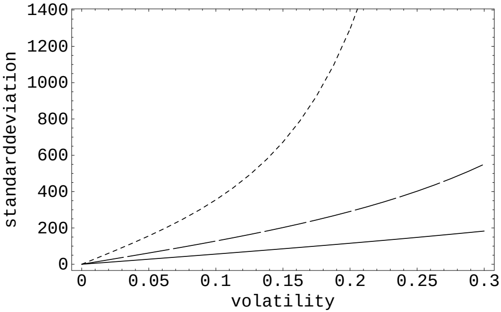
- Expected value and the variance of a simple CPPI are

$$E[V_t] = F_t + (V_0 - F_0)\exp\{(r + m(\mu - r))t\}$$

$$Var[V_t] = (V_0 - F_0)^2 \exp\{2(r + m(\mu - r))t\} \left(\exp\left\{m^2 \sigma^2 t\right\} - 1\right)$$

- Expected value independent of the volatility σ
- Standard deviation increases exponentially

Illustration: Standard deviation



Standard deviation of the final value of a simple CPPI with $V_0 = 1000$, $G = 800$, $T = 1$ and varying σ for $\mu = 0.1$, $r = 0.05$ and

$m = 2$ ($m = 4$, $m = 8$ respectively)

Conditions posed on discrete–time CPPI version

- Discrete–time version of the simple CPPI strategy satisfying the following three conditions
  - Value process converges in distribution to the value process of the continuous–time simple CPPI strategy
  - Self–financing
  - Non–negative asset exposure
- Implications
  - First condition implies that the cushion process of the discrete–time version converges to a lognormal process in distribution
  - The cushion process with respect to a discrete–time set of trading dates may also be negative

Definition of discrete–time CPPI version

- Notation

  - τ^n denote a sequence of equidistant refinements of the interval $[0, T]$, i.e.

$$\tau^n = \left\{ t_0^n = 0 < t_1^n < \cdots < t_{n-1}^n < t_n^n = T \right\}$$

- A strategy $\phi^\tau = (\eta^\tau, \beta^\tau)$ is called simple discrete–time CPPI if for $t \in]t_k, t_{k+1}]$ and $k = 0, \ldots, n-1$

$$\eta_t^\tau \quad := \quad \max\left\{ \frac{m\, C_{t_k}^\tau}{S_{t_k}}, 0 \right\}, \qquad \text{number of assets}$$

$$\beta_t^\tau \quad := \quad \frac{1}{B_{t_k}} \left(V_{t_k}^\tau - \eta_t^\tau S_{t_k} \right) \qquad \text{number of bonds}$$

Cushion process

Let $t_s := \min\left\{t_k \in \tau \middle| V^{\tau}_{t_k} - F_{t_k} \leq 0\right\}$

where $t_s = \infty$ if the minimum is not attained

Then

$$V^{\tau}_{t_{k+1}} - F_{t_{k+1}} = e^{r(t_{k+1} - \min\{t_s, t_{k+1}\})}\left(V^{\tau}_{t_0} - F_{t_0}\right) \prod_{i=1}^{\min\{s,k+1\}} \left(m\frac{S_{t_i}}{S_{t_{i-1}}} - (m-1)e^{r\frac{T}{n}}\right)$$

Events

Let

$$A_k := \left\{ \frac{S_{t_k}}{S_{t_{k-1}}} > \frac{m-1}{m} e^{r\frac{T}{n}} \right\}$$

for $k = 1, \ldots, n$, then it holds

$$\{t_s > t_i\} = \bigcap_{j=1}^{i} A_j$$

$$\text{and} \quad \{t_s = t_i\} = A_i^C \cap \left(\bigcap_{j=1}^{i-1} A_j \right) \quad \text{for} \quad i = 1, \ldots, n$$

Risk Measures

- Shortfall probability P^{SF}

$$P^{\text{SF}} := P\left(V_T^\tau \leq G\right) = P\left(V_T^\tau \leq F_T\right)$$

- Local shortfall probability P^{LSF}

$$P^{\text{LSF}} := P\left(V_{t_1}^\tau \leq F_{t_1} | V_{t_0}^\tau > F_{t_0}\right)$$

- Expected shortfall given default ESF

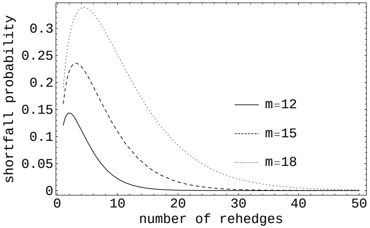
$$\text{ESF} := E\left[G - V_T^\tau | V_T^\tau \leq G\right]$$

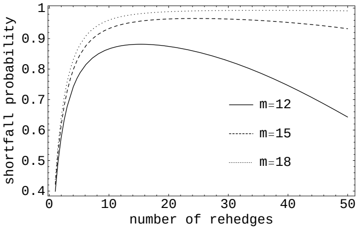
Shortfall probability

$$
\begin{aligned}
P^{\text{LSF}} &= \mathcal{N}(-d_2) \\
\text{where } d_2 &:= \frac{\ln\frac{m}{m-1} + (\mu - r)\frac{T}{n} - \frac{1}{2}\sigma^2\frac{T}{n}}{\sigma\sqrt{\frac{T}{n}}}
\end{aligned}
$$

$$
P^{\text{SF}} = 1 - \left(1 - P^{\text{LSF}}\right)^n
$$

Illustration: Shortfall probability ($T = 1$, $\mu = 0.085$ and $r = 0.05$)





$\sigma = 0.1$

$\sigma = 0.3$

Expected shortfall

- Expected final value

$$
\begin{aligned}
E\left[V_T^T\right] &= G + (V_0 - F_0)\left[E_1^n + e^{-r\frac{T}{n}} E_2 \frac{e^{rT} - E_1^n}{1 - E_1 e^{-r\frac{T}{n}}}\right] \\
\text{where} \quad E_1 &:= m e^{\mu\frac{T}{n}} \mathcal{N}(d_1) - e^{r\frac{T}{n}} (m-1) \mathcal{N}(d_2) \\
E_2 &:= e^{r\frac{T}{n}} \left[1 + m\left(e^{(\mu - r)\frac{T}{n}} - 1\right)\right] - E_1 .
\end{aligned}
$$

- Expected shortfall

$$
\mathrm{ESF} = \frac{(V_0 - F_0) e^{-r\frac{T}{n}} E_2 \frac{e^{rT} - E_1^n}{1 - E_1 e^{-r\frac{T}{n}}}}{P^{\mathrm{SF}}}
$$

Sensitivity of risk measures

Risk measures	Strategy parameter G	Strategy parameter m	Model parameter μ	Model parameter σ
Mean	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$
Stdv.	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$
P^{SF}	–	$\uparrow$	$\downarrow$	$\uparrow$
ESF	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$

Sensitivity analysis of risk measures symbol

- $\uparrow$ for monotonically increasing and
- $\downarrow$ for monotonically decreasing

Proportional transaction costs I

- Introduction of proportional transaction costs
- Proportionality factor is denoted by θ
- Intuition:
  Protection feature of the CPPI is based on a prespecified riskfree investment

⇒ Introduction of transaction costs must not change the number of risk free bonds which are prescribed by the CPPI method

⇒ Transaction costs are financed by a reduction of the asset exposure

⇒ Adjusting the cushion to the transaction costs gives

$$C_{t_{k+1}+} = C_{t_{k+1}} - \theta \left| mC_{t_{k+1}+} - mC_{t_k+} \frac{S_{t_{k+1}}}{S_{t_k}} \right|$$

Proportional transaction costs II

$$
\begin{aligned}
(i)\quad P^{\text{LSF,TA}} &= \mathcal{N}\left(-d_2^{\text{TA}}(\theta)\right) \\
d_2^{\text{TA}}(\theta) &:= \frac{\ln \frac{(1-\theta)m}{m-1} + (\mu - r)\frac{T}{n} - \frac{1}{2}\sigma^2 \frac{T}{n}}{\sigma\sqrt{\frac{T}{n}}} \\
(ii)\quad P^{\text{SF,TA}} &= 1 - \left(1 - P^{\text{LSF}}\right)^n \\
(iii)\quad \text{ESF}^{\text{TA}} &= \frac{\frac{V_0 - F_0}{1+\theta m} e^{-r\frac{T}{n}} E_2^{\text{TA}} \frac{e^{rT} - \left(E_1^{\text{TA}}\right)^n}{1 - E_1^{\text{TA}} e^{-r\frac{T}{n}}}}{P^{\text{SF,TA}}}
\end{aligned}
$$

Moments and risk measures

Parameter constellation:

$\mu = 0.085$, $\sigma = 0.1$ (0.2 or 0.3, respectively), $r = 0.05$, $T = 1$ and $V_0 = G = 1000$

n	m	Mean	Stdv. Dev.	*SFP*	*ESF*
12	10	1072.43 (1073.22)	88.56 (368.16)	**0.0011 (0.3265)**	3.72 (14.87)
36	10	1072.65 (1072.67)	92.95 (463.935)	0.0000 (0.0268)	1.37 (5.00)
60	10	1072.69 (1072.69)	93.90 (489.08)	0.0000 (0.0013)	0.00 (3.13)
∞	10	1072.76 (1072.76)	95.37 (532.66)	0.0000 (0.0000)	0.00 (0.00)

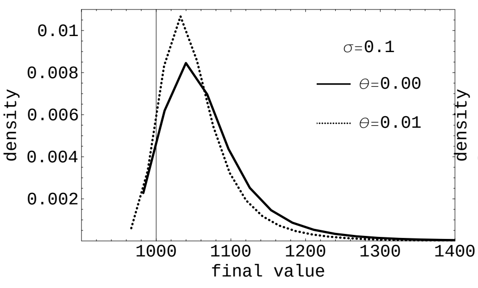
Moments and risk measures for $\sigma = 0.1$ ($\sigma = 0.2$ respectively)

Risk profile

	$\theta = 0.00$		$\theta = 0.01$	
n	m	ESF	m	ESF
12	11.843 (6.065)	5.313 (4.478)	10.684 (5.772)	4.116 (3.925)
36	18.146 (9.234)	5.149 (4.190)	15.490 (8.531)	2.500 (2.824)
60	22.336 (11.335)	5.243 (4.121)	18.409 (10.274)	1.603 (2.088)

m for an implied shortfall probability of 0.01 and $\sigma = 0.1$ ($\sigma = 0.2$)

Distribution of final CPPI value



Conclusion

- CPPI strategies are common in hedge funds and retail products ($\rightarrow$ meaningful risk management and pricing must take into account the gap risk)
- Introduction of trading restrictions is one possibility to model a gap risk in the sense that a CPPI strategy can not be adjusted adequately
- The analysis of the risk measures of a discrete–time CPPI strategy poses various problems which are to be considered
- Basically, it is necessary to check the associated risk measures and to determine whether the strategy is still effective in terms of portfolio protection