

Law of the Minimal Price

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Law of One Price

“All replicating portfolios of a payoff have the same price!”

Debreu (1959), Sharpe (1964), Lintner (1965),

Merton (1973a, 1973b), Ross (1976), Harrison & Kreps (1979),

Cochrane (2001), . . .

will be, in general, violated under the benchmark approach.

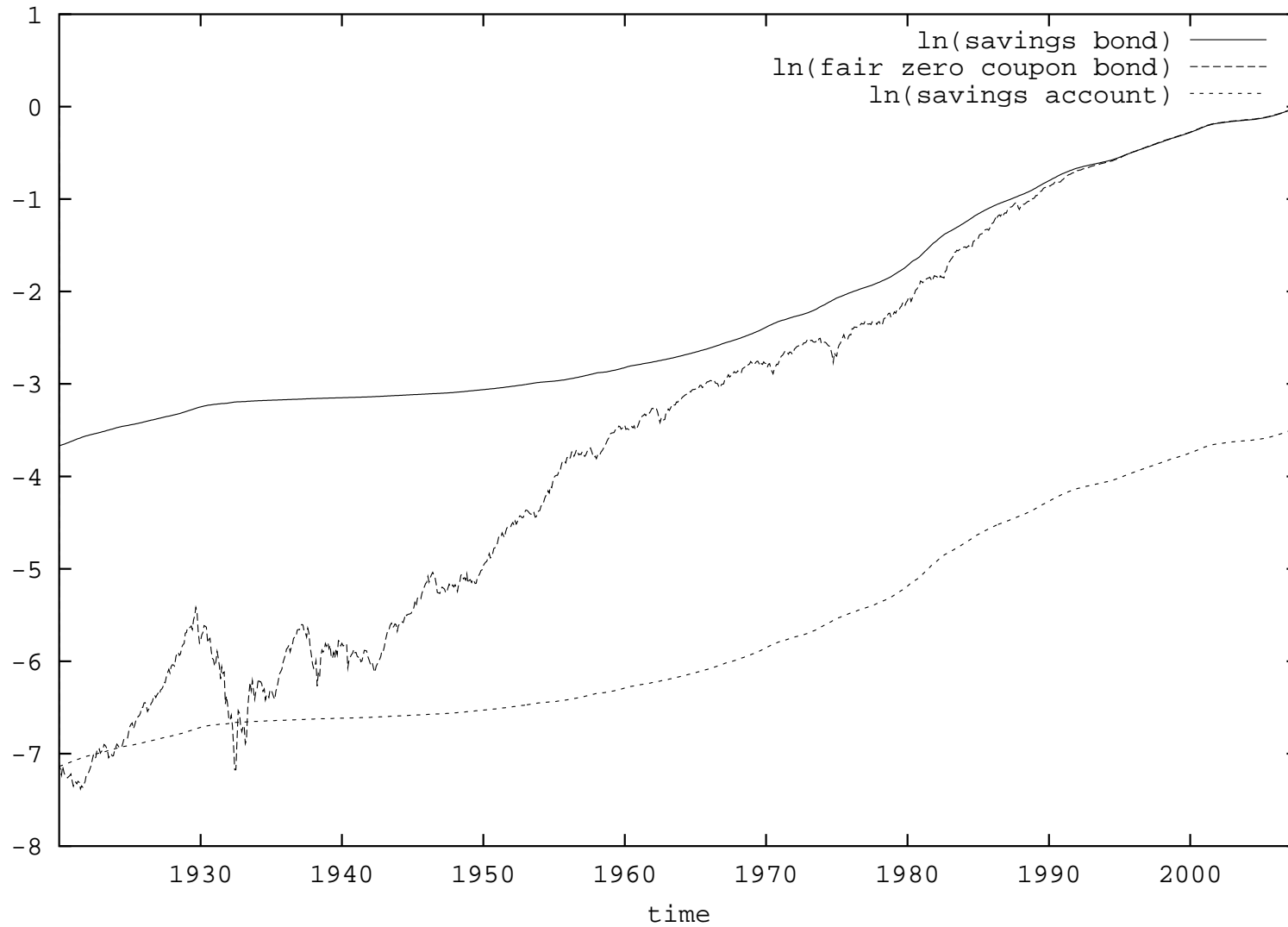


Figure 1: Logarithms of savings bond, fair zero coupon bond and savings account.

Two Asset Market Example

- trading times $t_i = i h, h > 0$
- **risky asset** S^1 (S&P500 accumulation index)

$$\text{asset ratio } A_{t_i, h}^1 = \frac{S_{t_i+h}^1}{S_{t_i}^1} \in (0, \infty)$$

can reach **any** finite strictly positive value

- **savings account** S^0 (US savings account)

$$\text{asset ratio } A_{t_i, h}^0 = \frac{S_{t_i+h}^0}{S_{t_i}^0} > 0$$

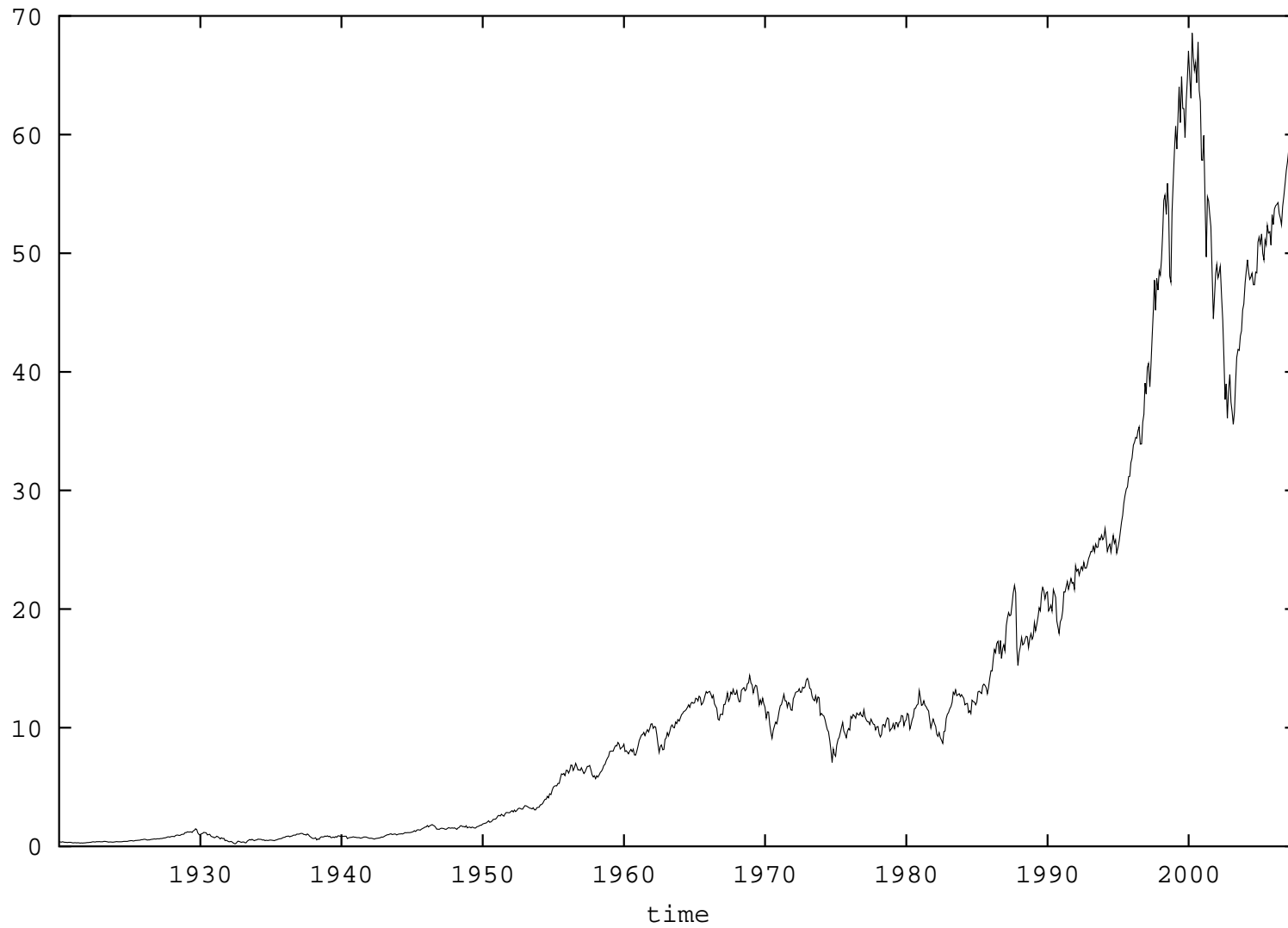


Figure 2: Discounted S&P500.

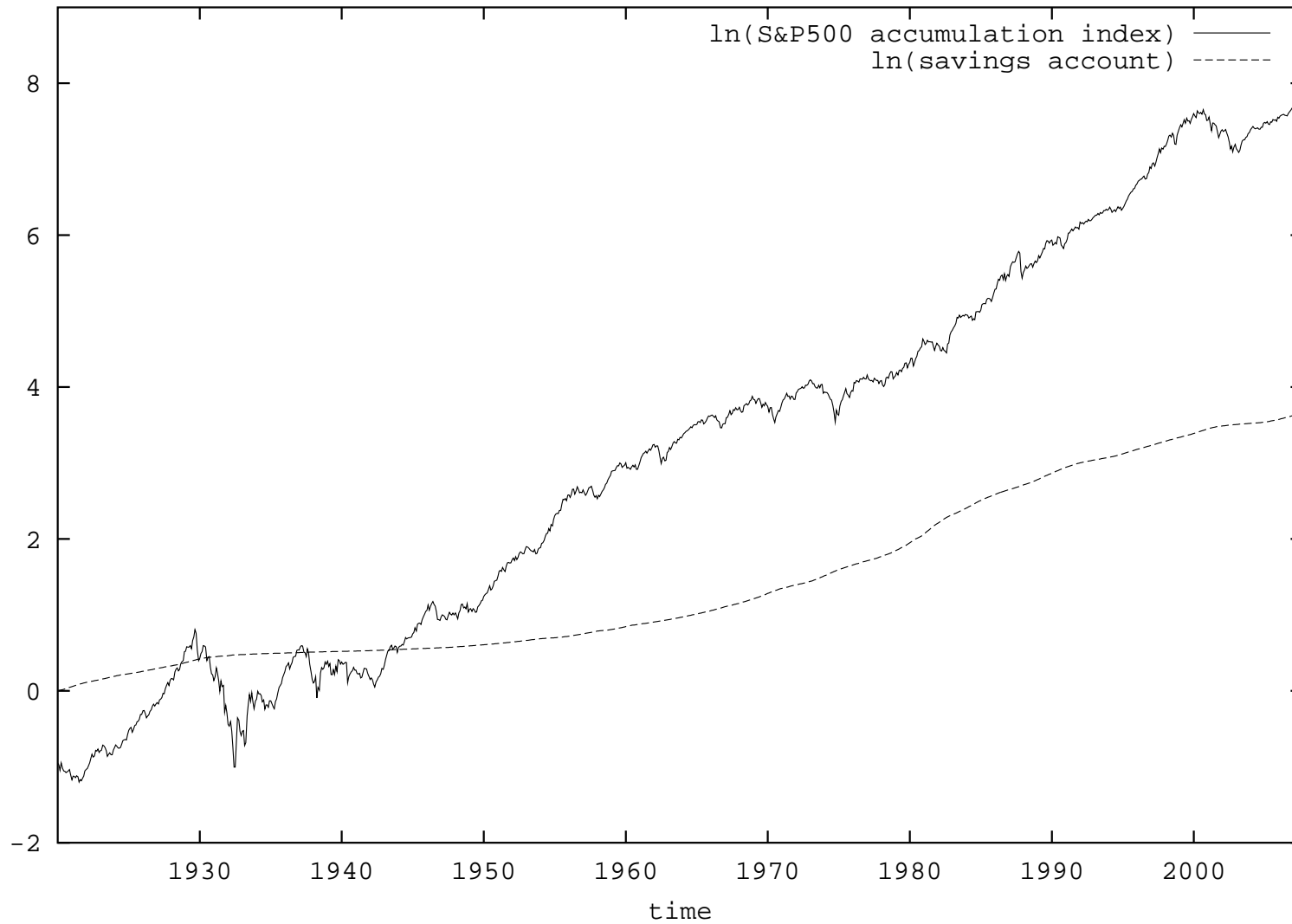


Figure 3: $\ln(\text{S\&P500 accumulation index})$ and $\ln(\text{savings account})$.

- portfolio S^δ

strict positivity of $S^\delta \iff \pi_{\delta,t_i}^0 = \delta_{t_i}^0 \frac{S_{t_i}^0}{S_{t_i}^\delta} \in [0, 1]$

$$A_{t_i,h}^\delta = \frac{S_{t_i+h}^\delta}{S_{t_i}^\delta} = \pi_{\delta,t_i}^0 A_{t_i,h}^0 + \left(1 - \pi_{\delta,t_i}^0\right) A_{t_i,h}^1$$

- find **best performing** strictly positive portfolio

- **expected growth**

$$g_{t_i,h}^\delta = E_{t_i} \left(\ln \left(A_{t_i,h}^\delta \right) \right)$$

$$\frac{\partial g_{t_i,h}^\delta}{\partial \pi_{\delta,t_i}^0} = E_{t_i} \left(\frac{A_{t_i,h}^0 - A_{t_i,h}^1}{A_{t_i,h}^\delta} \right)$$

second derivative negative

\implies one genuine maximum at $\pi_{\underline{\delta},t_i}^0 \in (-\infty, \infty)$ for

$$\frac{\partial g_{t_i,h}^\delta}{\partial \pi_{\delta,t_i}^0} = 0$$

- **Three cases:**

(i) $\pi_{\underline{\delta}, t_i}^0 \in [0, 1]$ classical case

(ii) $\pi_{\underline{\delta}, t_i}^0 < 0$ savings account performs poorly, risk premium

\implies **index** is best performing portfolio

(iii) $\pi_{\underline{\delta}, t_i}^0 > 1$ index performs poorly

\implies **savings account** is best performing portfolio

- **classical case (i):** $\pi_{\underline{\delta}, t_i}^0 \in [0, 1]$

\implies

$$E_{t_i} \left(\frac{A_{t_i, h}^1}{A_{t_i, h}^{\underline{\delta}}} \right) = E_{t_i} \left(\frac{A_{t_i, h}^0}{A_{t_i, h}^{\underline{\delta}}} \right) = 1$$

\implies genuine maximum $g_{t_i, h}^{\delta^*} = g_{t_i, h}^{\underline{\delta}}$

$S_{t_i}^0$ and $S_{t_i}^1$ are constituents of $S_{t_i}^{\delta^*}$

$\frac{S_{t_i}^0}{S_{t_i}^{\delta^*}}$ and $\frac{S_{t_i}^1}{S_{t_i}^{\delta^*}}$ are martingales $\hat{=}$ fair

\implies all benchmarked portfolios are fair

\implies **Law of One Price holds**

- other cases

rewrite first order condition

$$\frac{\partial g_{t_i, h}^\delta}{\partial \pi_{\delta, t_i}^0} = E_{t_i} \left(Q_{t_i, h} - \pi_{\delta, t_i}^0 \frac{(Q_{t_i, h})^2}{1 + \pi_{\delta, t_i}^0 Q_{t_i, h}} \right) = 0$$

with

$$Q_{t, h} = \frac{\frac{S_{t+h}^0}{S_{t+h}^1}}{\frac{S_t^0}{S_t^1}} - 1$$

\implies

$$\pi_{\underline{\delta}, t}^0 = \frac{\lim_{h \rightarrow 0} \frac{1}{h} E_t(Q_{t, h})}{\lim_{h \rightarrow 0} \frac{1}{h} E_t \left(\frac{(Q_{t, h})^2}{1 + \pi_{\underline{\delta}, t}^0 Q_{t, h}} \right)}$$

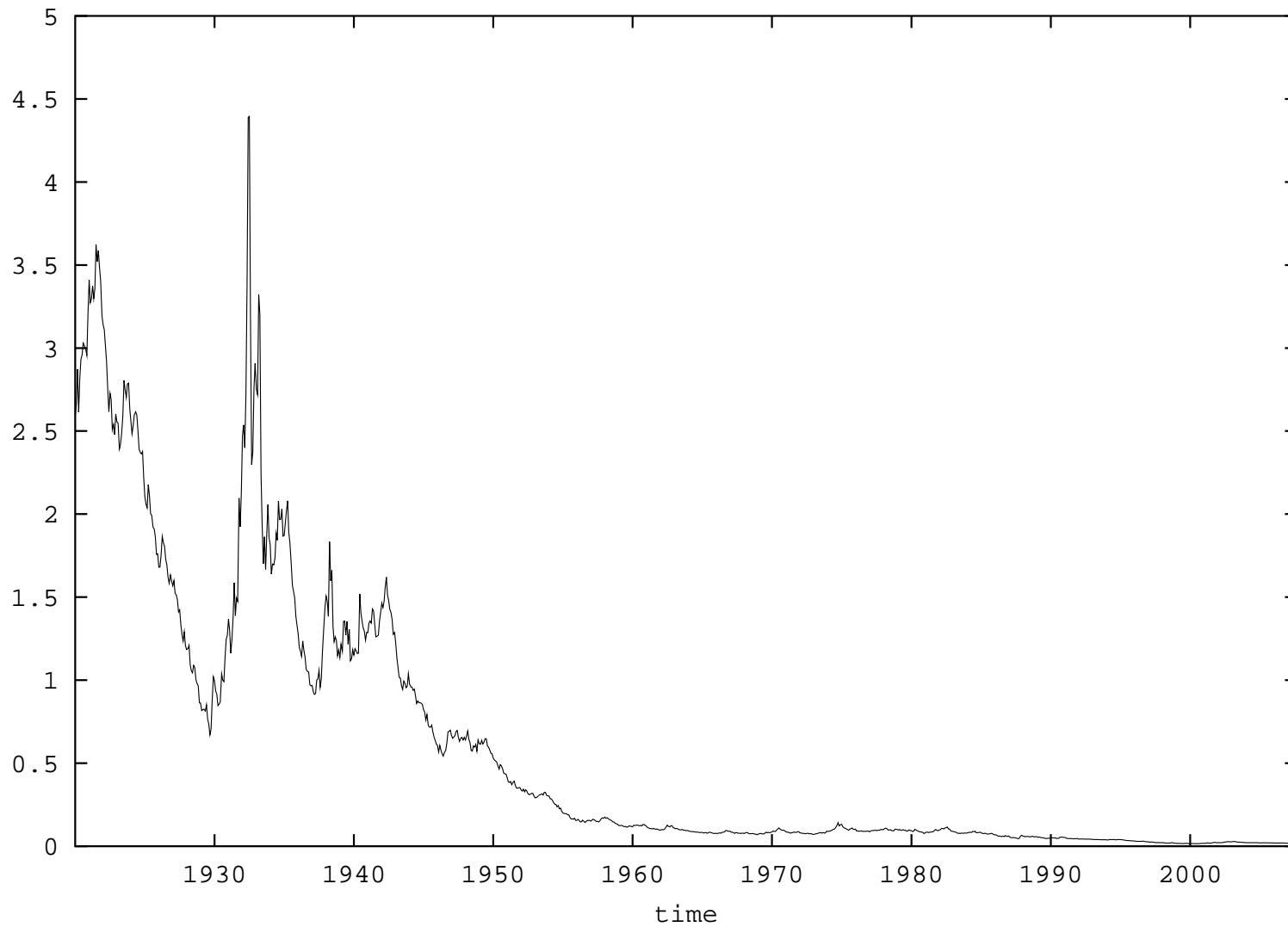


Figure 4: US-benchmarked savings account.

- annualized returns of benchmarked savings account
 $n = 1052$

$$\hat{\mu} = \frac{1}{n+1} \sum_{i=0}^n \frac{1}{h} Q_{t_i, h} \approx -0.0396$$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=0}^n \frac{1}{h} (Q_{t_i, h})^2} \approx 0.19$$

negative fraction $\pi_{\underline{\delta}, t}^0 = \frac{\hat{\mu}}{\hat{\sigma}^2} \approx -1.1$

\implies savings account **unlikely** to be fair

Extreme Maturity Bond

- need model with downward trending $\frac{S_t^0}{S_t^{\delta^*}}$
to reflect reality

- assume short rate deterministic

- **savings bond**

$$P^*(t, T) = \frac{S_t^0}{S_T^0}$$

- select index S^1 as numeraire portfolio S^{δ^*}

- benchmarked fair zero coupon bond

$$\hat{P}(t, T) = \frac{P(t, T)}{S_t^{\delta_*}} = E_t \left(\frac{1}{S_T^{\delta_*}} \right)$$

martingale

\implies real world pricing formula

$$P(t, T) = S_t^{\delta_*} E_t \left(\frac{1}{S_T^{\delta_*}} \right)$$

- **discounted numeraire portfolio**

$$\bar{S}_t^{\delta_*} = \frac{S_t^{\delta_*}}{S_t^0}$$

for continuous market generally satisfies SDE

$$d\bar{S}_t^{\delta_*} = \alpha_t dt + \sqrt{\bar{S}_t^{\delta_*}} \alpha_t dW_t$$

is time transformed squared Bessel process

- model the drift of discounted index as

$$\alpha_t = \frac{\alpha}{\eta} \exp\{\eta t\}$$

\implies **Minimal Market Model**

MMM, see Pl. & Heath (2006)

- **net growth rate**

$\eta \approx 0.0511$ with R^2 of **0.88**

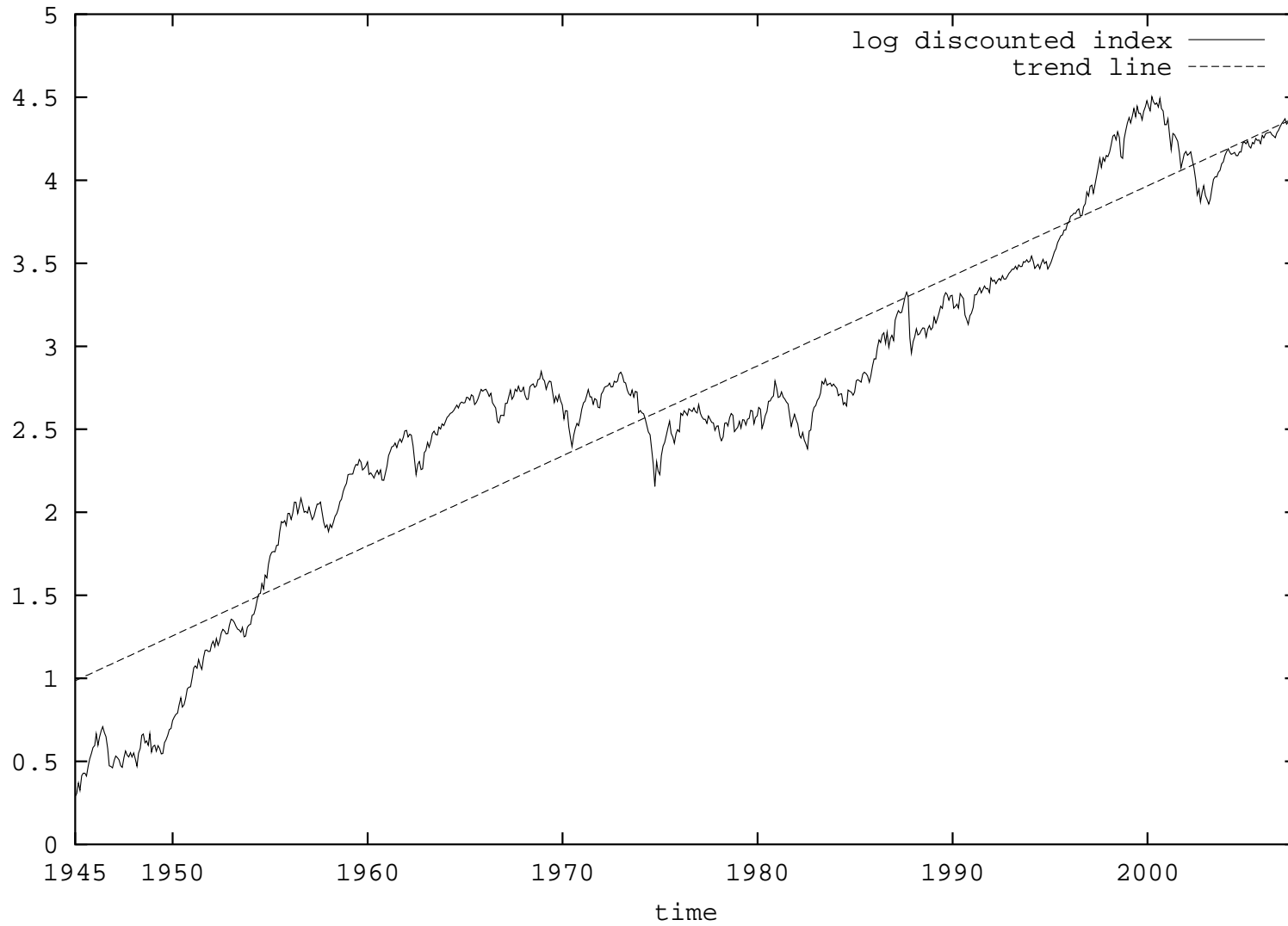


Figure 5: Logarithm of discounted index.

- **normalized index**

$$Y_t = \frac{\bar{S}_t^{\delta_*}}{\alpha_t}$$

$$dY_t = (1 - \eta Y_t) dt + \sqrt{Y_t} dW_t$$

- **quadratic variation of $\sqrt{Y_t}$**

$$d\sqrt{Y_t} = \dots + \frac{1}{2} dW_t$$

$$V_{t,h} = \sum_{\ell=1}^{i_t} \left(\sqrt{Y_{t\ell}} - \sqrt{Y_{t\ell-1}} \right)^2 \approx \left[\sqrt{Y} \right]_t = \frac{t}{4}$$

- **scaling parameter** $\alpha \approx 0.01429$ with R^2 of 0.995

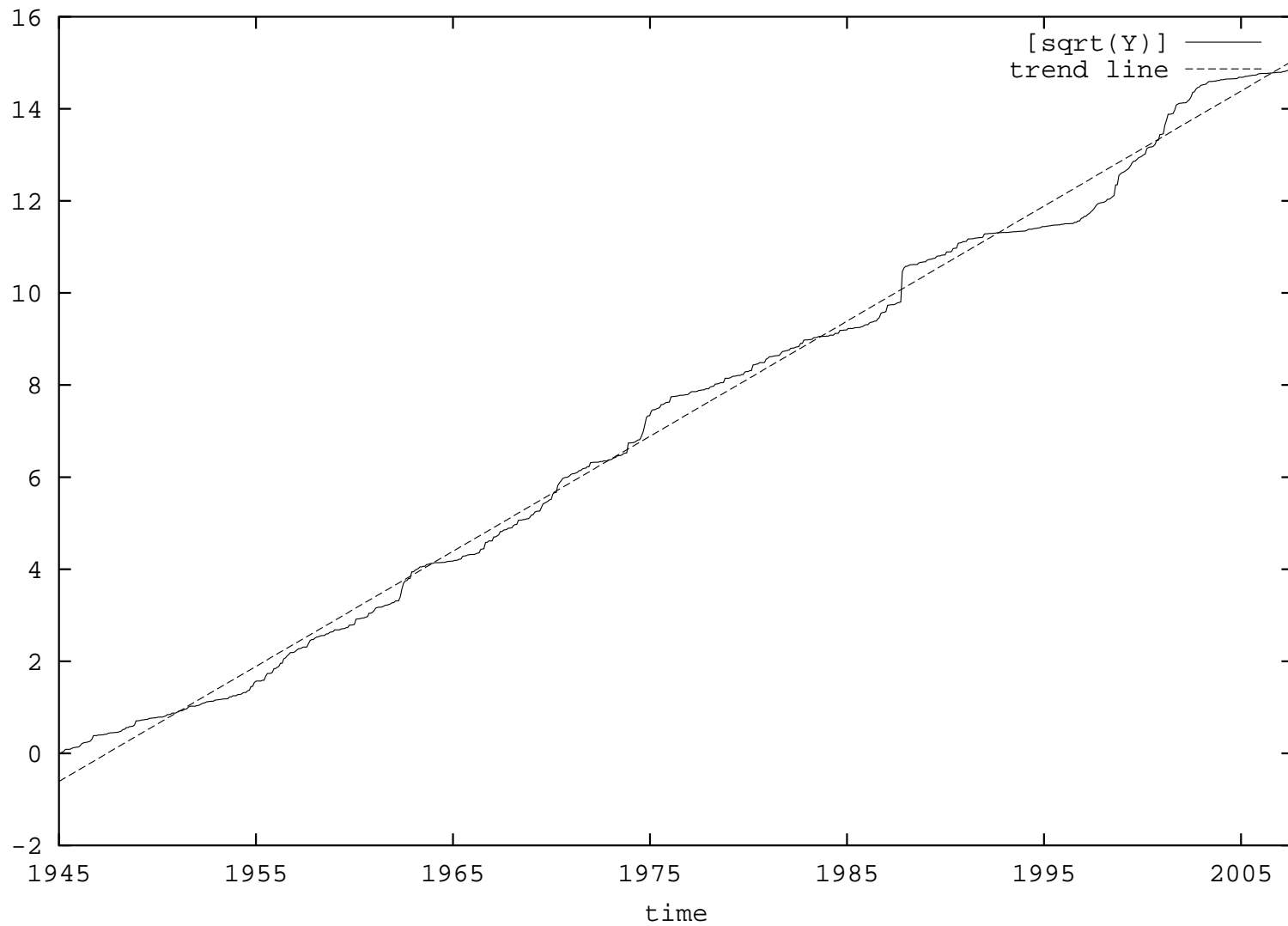


Figure 6: Quadratic Variation of $\sqrt{Y_t}$.

- **fair zero coupon bond (MMM)**

$$P(t, T) = P^*(t, T) \left(1 - \exp \left\{ -\frac{2 \eta \bar{S}_t^{\delta^*}}{\alpha (\exp\{\eta T\} - \exp\{\eta t\})} \right\} \right)$$

- **initial prices in 1920:**

$$P^*(0, T) = 0.025496$$

$$P(0, T) = 0.000795$$

$$\frac{P(0, T)}{P^*(0, T)} < 0.0312 \quad \implies \quad \mathbf{3.12\%}$$

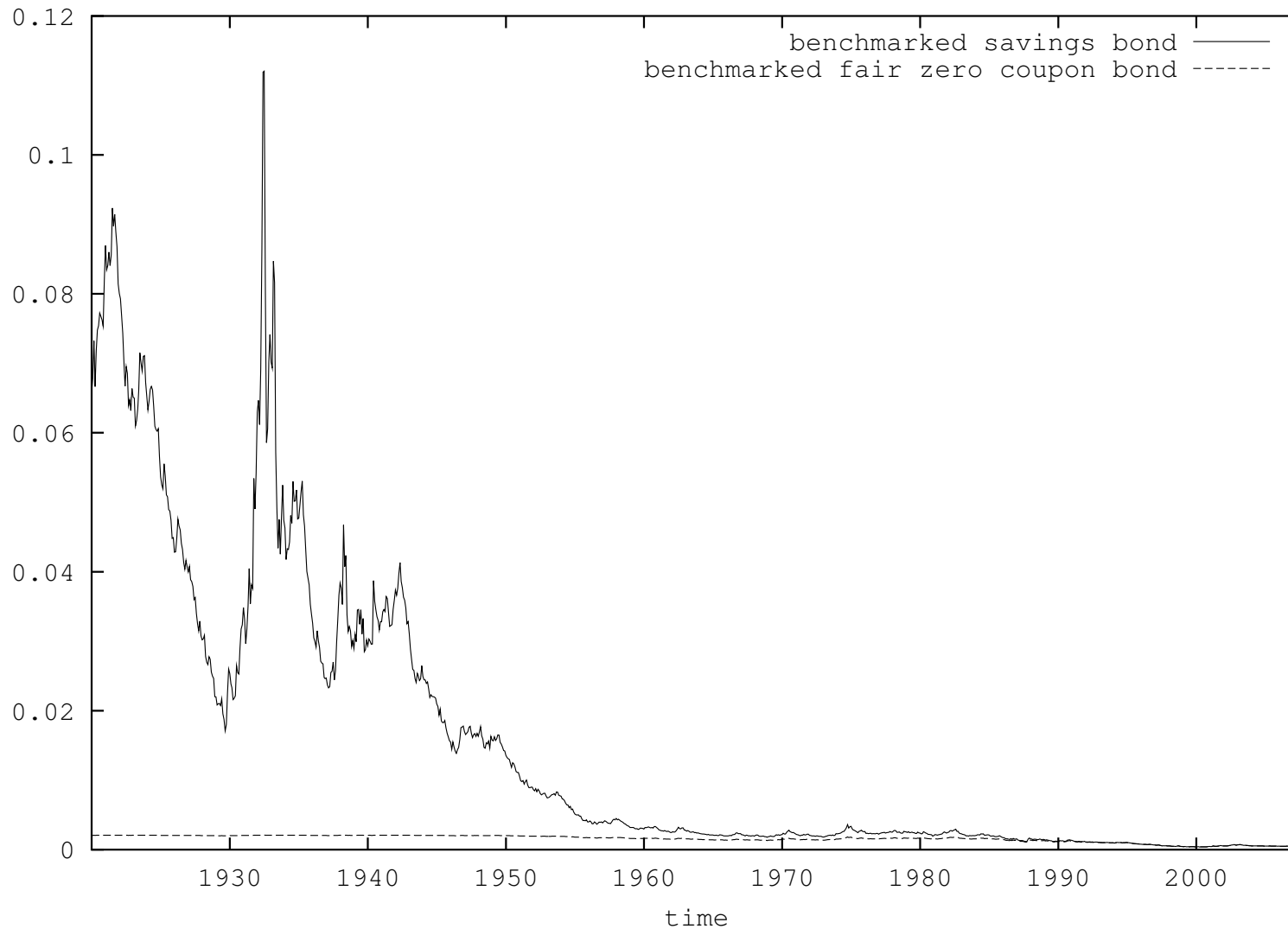


Figure 7: Benchmarking savings bond and benchmarking fair zero coupon bond.

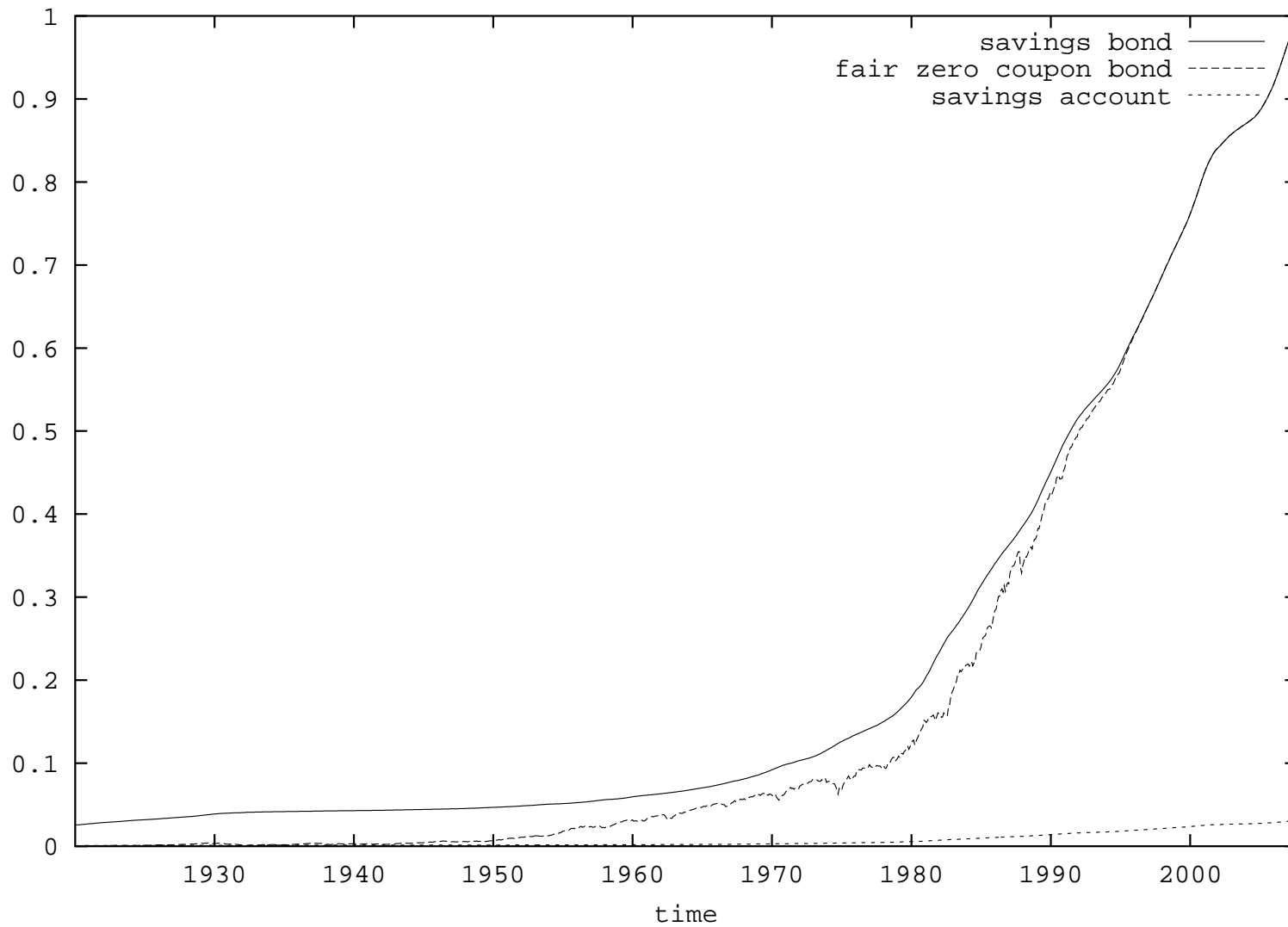


Figure 8: Savings bond, fair zero coupon bond and savings account.

- **self-financing hedge portfolio** hedge ratio

$$\begin{aligned}
 \delta_t^* &= \frac{\partial P(t, T)}{\partial \bar{S}_t^{\delta^*}} \\
 &= P^*(t, T) \exp \left\{ \frac{-2 \eta \bar{S}_t^{\delta^*}}{\alpha (\exp\{\eta T\} - \exp\{\eta t\})} \right\} \\
 &\quad \times \frac{2 \eta}{\alpha (\exp\{\eta T\} - \exp\{\eta t\})}
 \end{aligned}$$

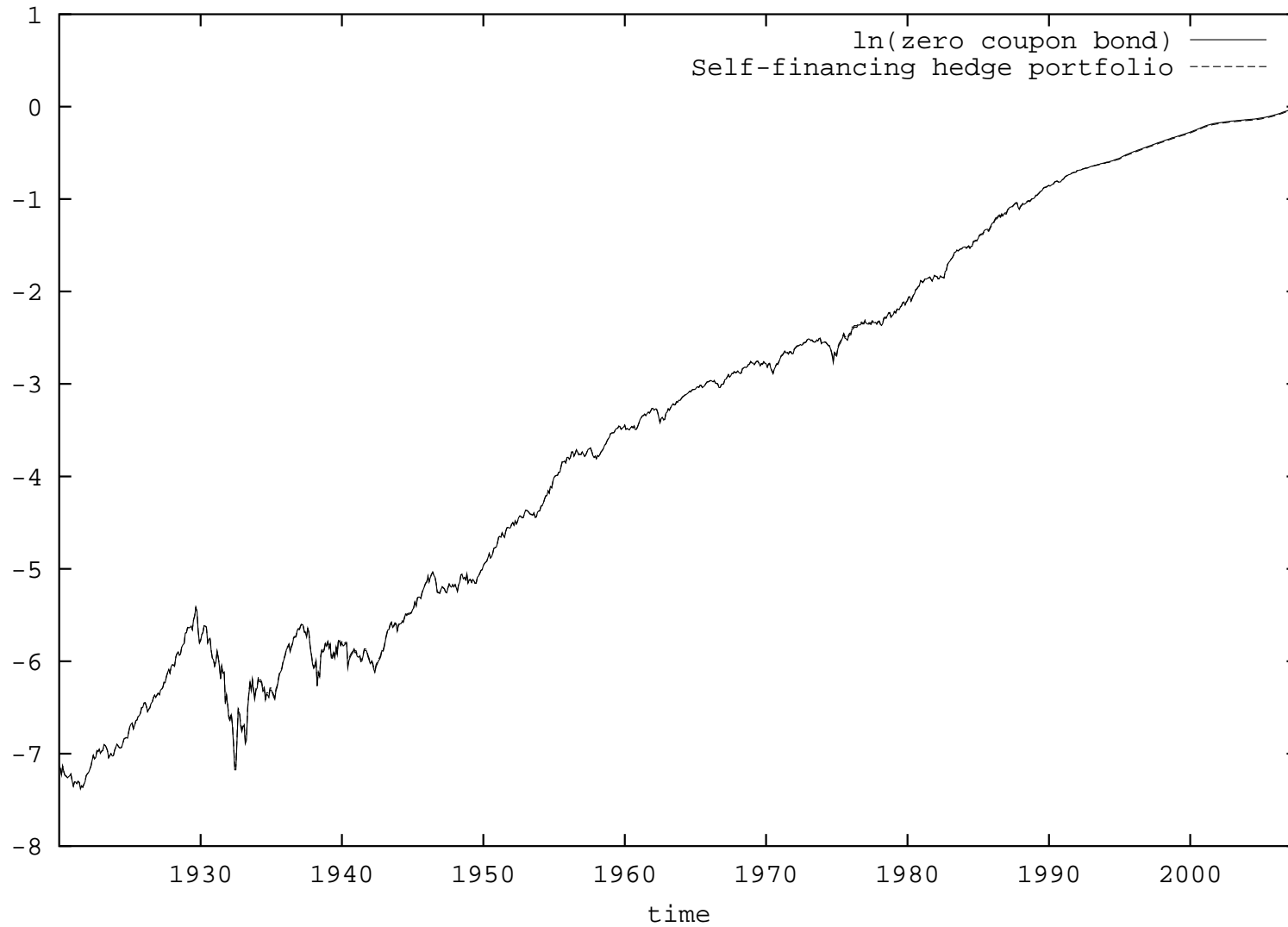


Figure 9: Logarithm of zero coupon bond and self-financing hedge portfolio.

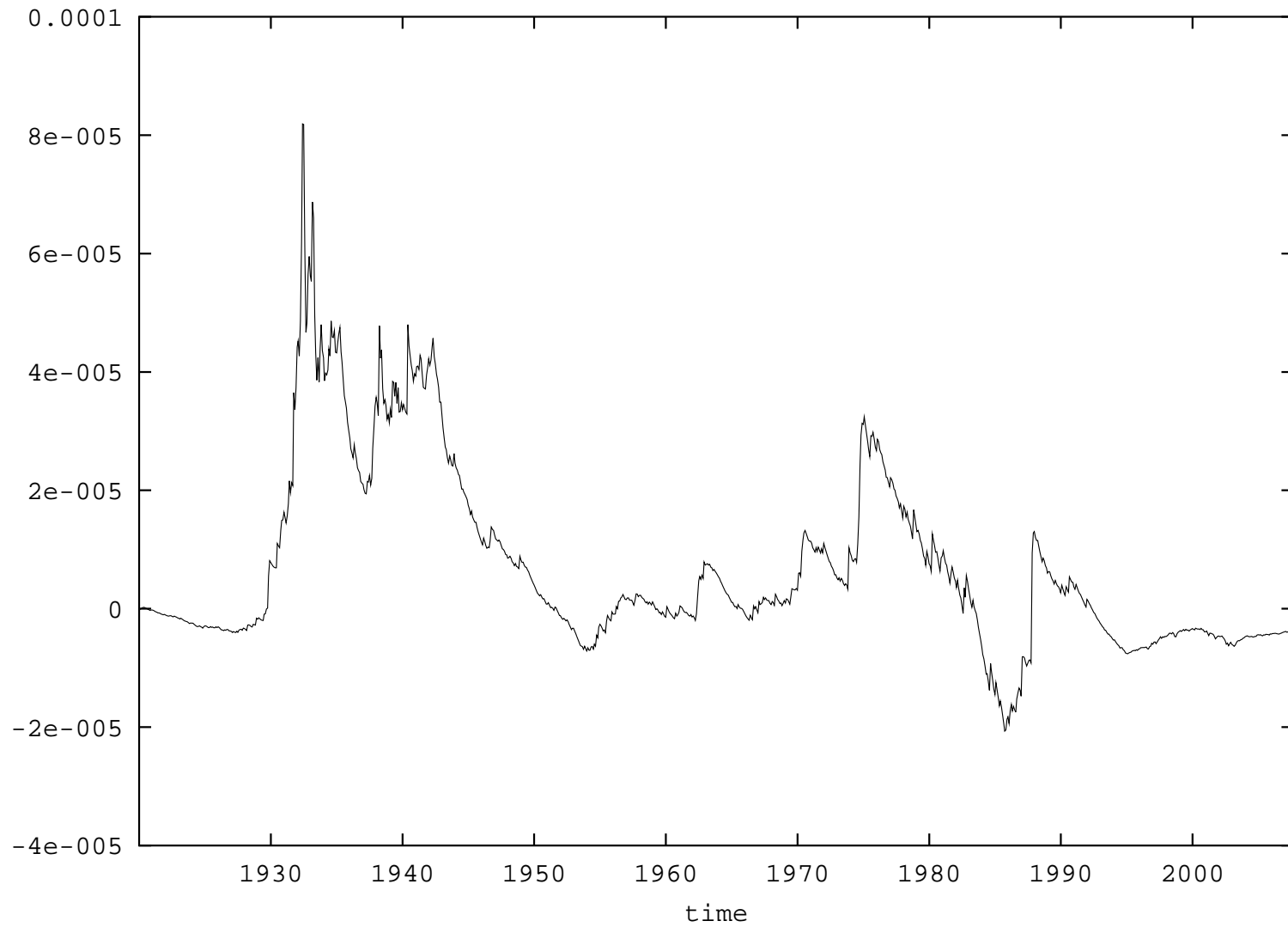


Figure 10: Benchmarked P&L.

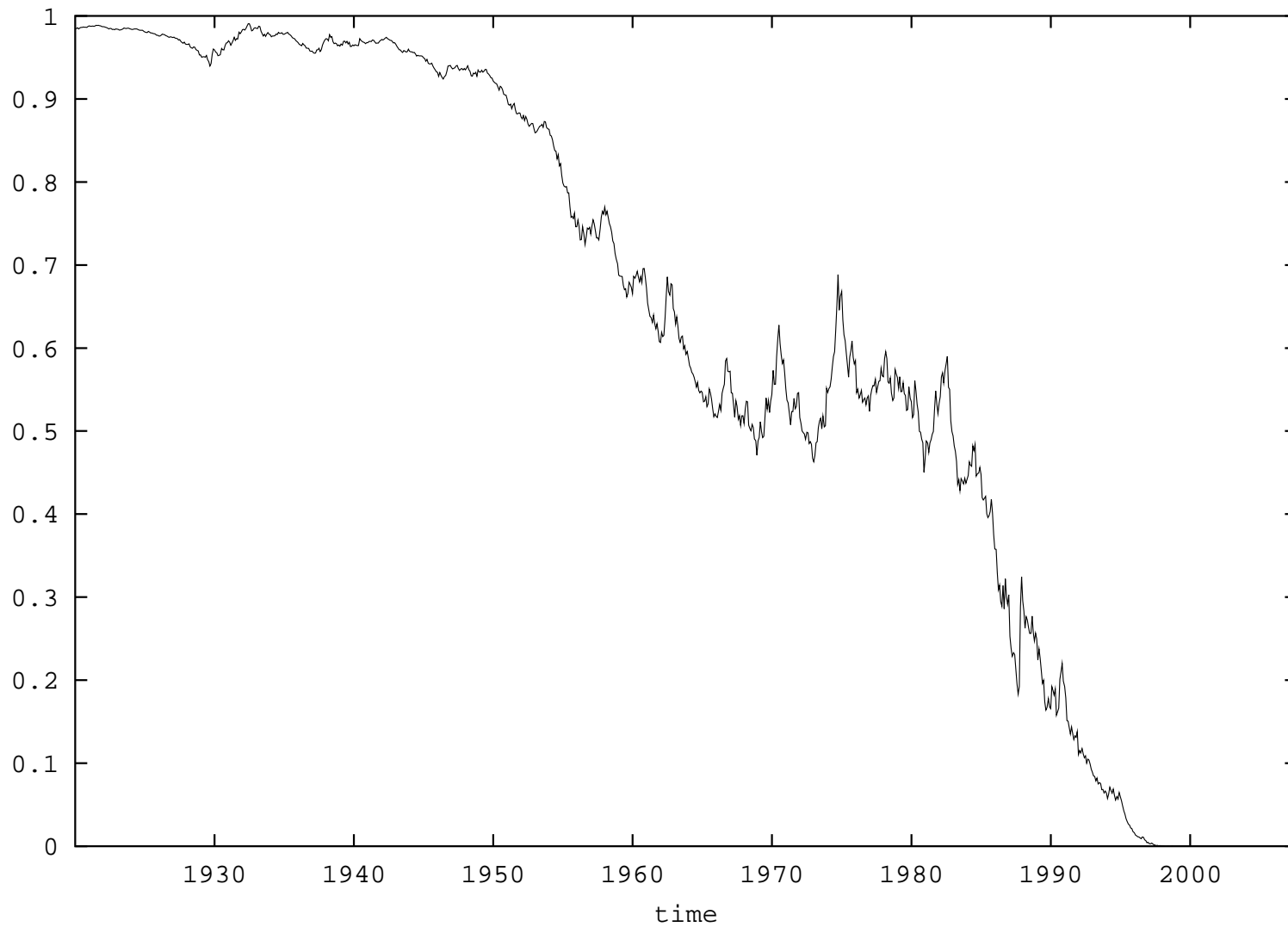


Figure 11: Fraction invested in the index.

Financial Market

- j th primary security account

$$S_t^j$$

$$j \in \{0, 1, \dots, d\}$$

- savings account

$$S_t^0$$

$$t \geq 0$$

- **strategy**

$$\delta = \{\delta_t = (\delta_t^0, \delta_t^1, \dots, \delta_t^d)^\top, t \geq 0\}$$

predictable

- **portfolio**

$$S_t^\delta = \sum_{j=0}^d \delta_t^j S_t^j$$

- **self-financing**

$$dS_t^\delta = \sum_{j=0}^d \delta_t^j dS_t^j$$

Numeraire Portfolio

Definition 1 $S^{\delta_*} \in \mathcal{V}_x^+$ **numeraire portfolio** if

$$E_t \left(\frac{\frac{S_{t+h}^\delta}{S_{t+h}^{\delta_*}}}{\frac{S_t^\delta}{S_t^{\delta_*}}} - 1 \right) \leq 0$$

for all nonnegative S^δ and $t, h \in [0, \infty)$.

- S^{δ_*} “best” performing portfolio

Long (1990), Becherer (2001), Pl. (2002, 2006),

Bühlmann & Pl. (2003), Goll & Kallsen (2003), Karatzas & Kardaras (2007)

Main Assumption of the Benchmark Approach

Assumption 2 *There exists a numeraire portfolio $S^{\delta*} \in \mathcal{V}_x^+$.*

Supermartingale Property

- benchmarked value

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}}$$

Corollary 3 For nonnegative S^δ

$$\hat{S}_t^\delta \geq E_t \left(\hat{S}_s^\delta \right)$$

$$0 \leq t \leq s < \infty$$

nonnegative \hat{S}^δ supermartingale

Definition 4 Price is **fair** if, when benchmarked, forms martingale.

$$\hat{S}_t^\delta = E_t \left(\hat{S}_s^\delta \right)$$

for $0 \leq t \leq s < \infty$.

Lemma 5 *The minimal nonnegative supermartingale that reaches a given benchmarked contingent claim is a martingale.*

see Pl. & Heath (2006)

Law of the Minimal Price

Theorem 6 *If a fair portfolio replicates a nonnegative payoff, then this represents the minimal replicating portfolio.*

- least expensive
- minimal hedge
- economically correct price in a competitive market

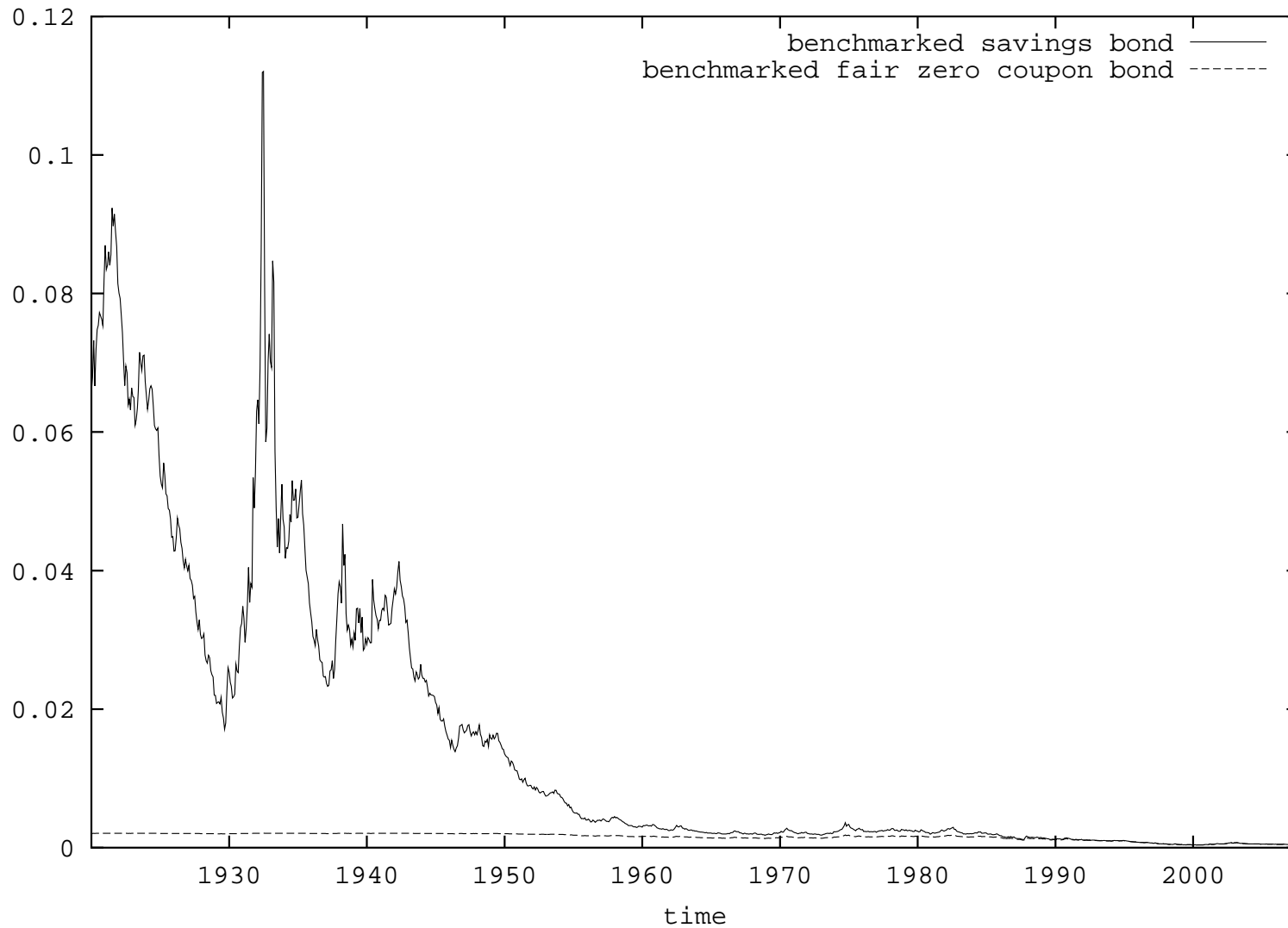


Figure 12: Benchmarking savings bond and benchmarking fair zero coupon bond.

- **claim**

H_T

$$E_0 \left(\frac{H_T}{S_T^{\delta_*}} \right) < \infty$$

Corollary 7

Minimal price for replicable H_T is given by

real world pricing formula

$$S_t^{\delta_H} = S_t^{\delta_*} E_t \left(\frac{H_T}{S_T^{\delta_*}} \right) .$$

- **normalized benchmarked savings account**

$$\Lambda_T = \frac{\hat{S}_T^0}{\hat{S}_0^0}$$

$$1 = \Lambda_0 \geq E_0(\Lambda_T)$$

- **real world pricing formula** \implies

$$S_0^{\delta_H} = E_0 \left(\Lambda_T \frac{S_0^0}{S_T^0} H_T \right)$$

\implies

$$S_0^{\delta_H} \leq \frac{E_0 \left(\Lambda_T \frac{S_0^0}{S_T^0} H_T \right)}{E_0(\Lambda_T)}$$

similar for any numeraire

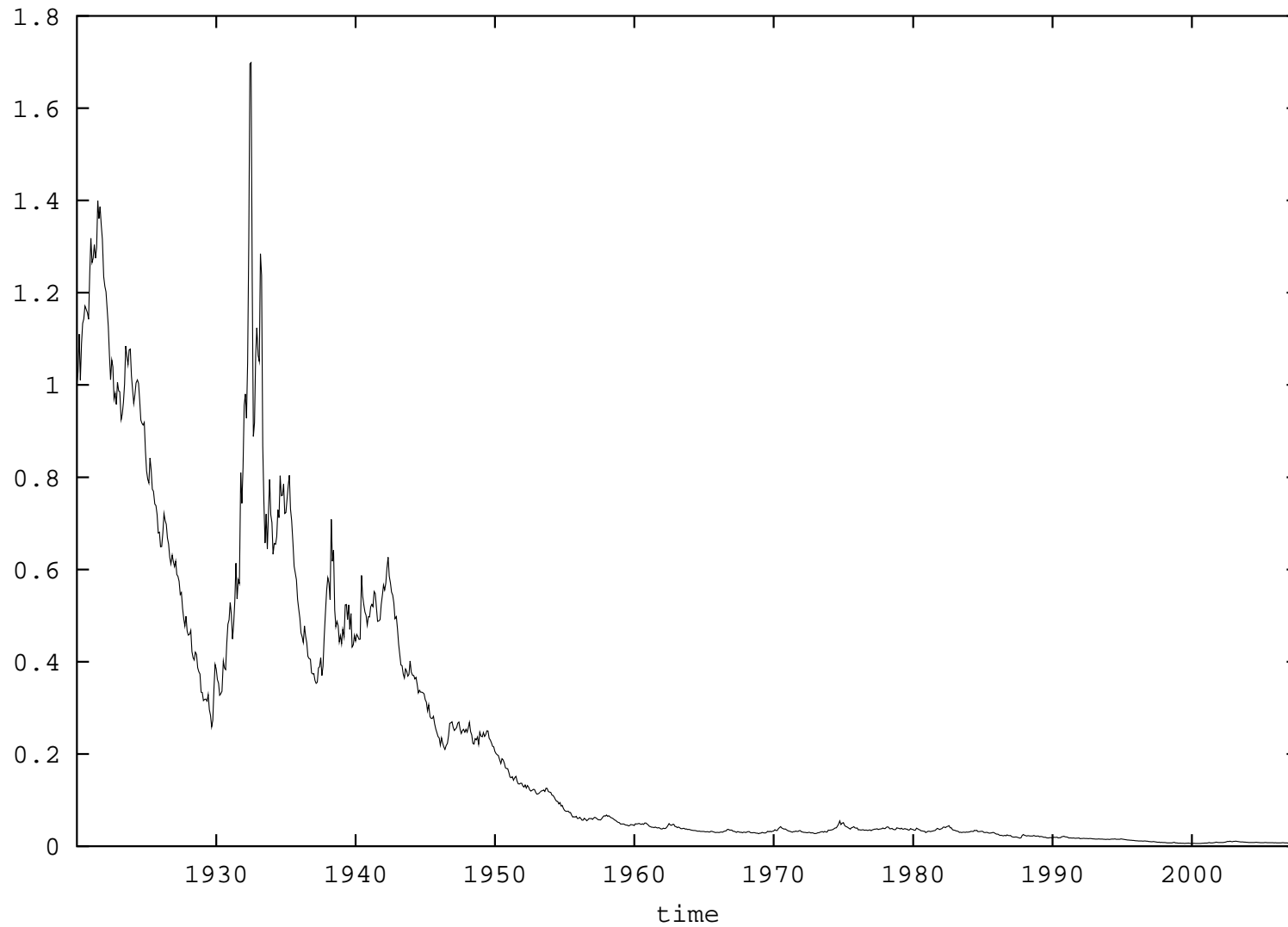


Figure 13: Candidate Radon-Nikodym derivative of hypothetical risk neutral measure of real market.

- special case when **savings account is fair**:

$$\implies \Lambda_T = \frac{dQ}{dP} \text{ forms martingale; } E_0(\Lambda_T) = 1;$$

equivalent risk neutral probability measure Q exists;

Bayes' formula \implies

risk neutral pricing formula

$$S_0^{\delta_H} = E_0^Q \left(\frac{S_0^0}{S_T^0} H_T \right)$$

Harrison & Kreps (1979), Ingersoll (1987),

Constatinides (1992), Duffie (2001), Cochrane (2001), ...

- otherwise “risk neutral price” \geq real world price

- long term growth rate

$$g^\delta = \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \left(\frac{S_t^\delta}{S_0^\delta} \right)$$

Theorem 8 For $S^\delta \in \mathcal{V}_x^+$

$$g^\delta \leq g^{\delta*}.$$

- “pathwise best” in the long run

Karatzas & Shreve (1998), Pl. (2004),

Karatzas & Kardaras (2007)

Pl. (2004)

Definition 9 *Nonnegative portfolio S^δ **outperforms systematically** $S^{\tilde{\delta}}$*
if

$$(i) \quad S_0^\delta = S_0^{\tilde{\delta}};$$

$$(ii) \quad P \left(S_t^\delta \geq S_t^{\tilde{\delta}} \right) = 1$$

$$(iii) \quad P \left(S_t^\delta > S_t^{\tilde{\delta}} \right) > 0.$$

- “relative arbitrage” Fernholz & Karatzas (2005)

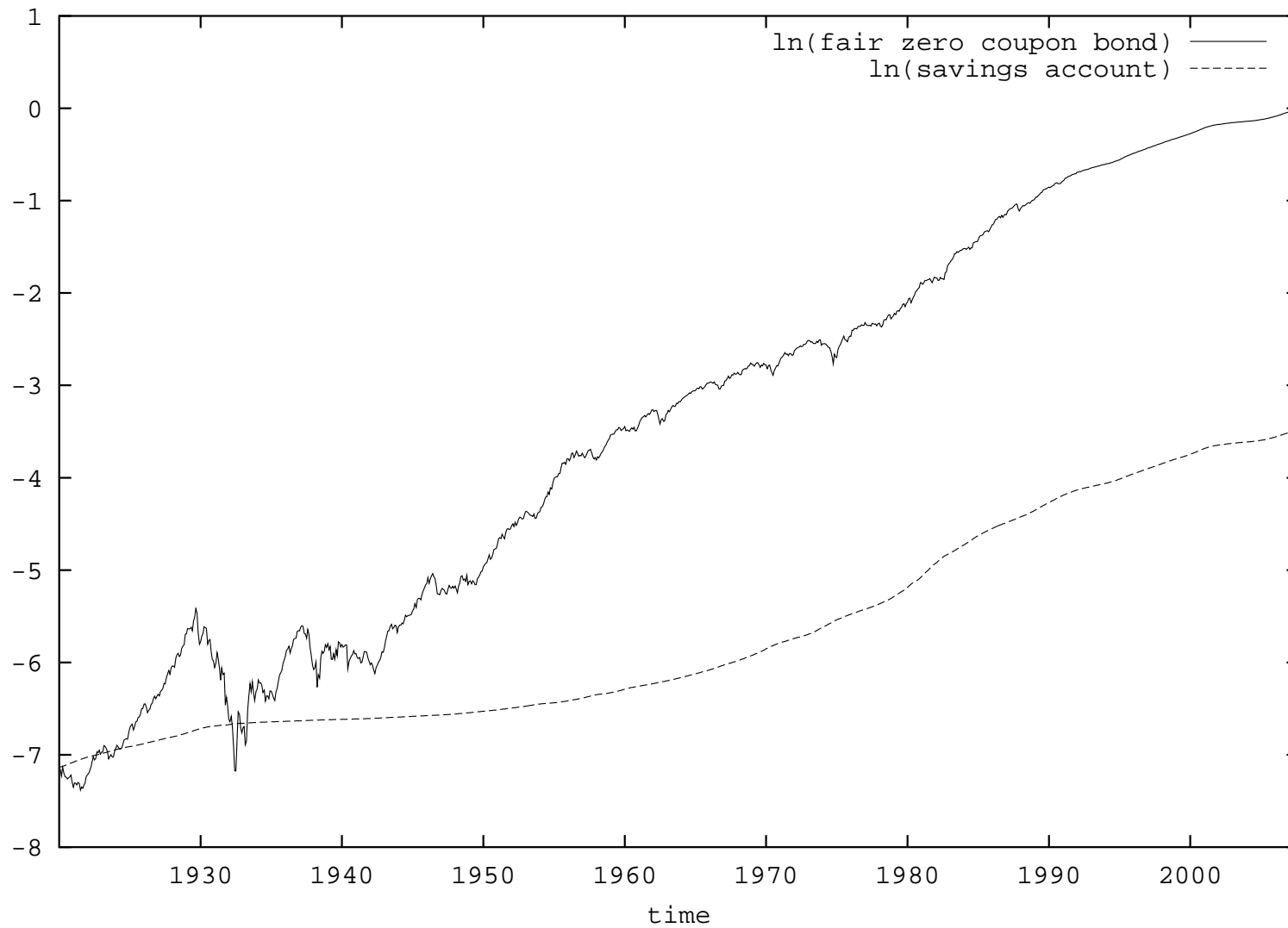


Figure 14: Logarithms of fair zero coupon bond and savings account.

Theorem 10 *Numeraire portfolio cannot be outperformed systematically.*

- **portfolio ratio**

$$A_{t,h}^{\delta} = \frac{S_{t+h}^{\delta}}{S_t^{\delta}}$$

- **expected growth**

$$g_{t,h}^{\delta} = E_t \left(\ln \left(A_{t,h}^{\delta} \right) \right)$$

$$t, h \geq 0$$

- **derivative of expected growth**

$$S^{\underline{\delta}} \in \mathcal{V}_x^+$$

$S^{\underline{\delta}}$ nonnegative

S^{δ_ε} perturbed $S^{\underline{\delta}}$

$$A_{t,h}^{\delta_\varepsilon} = \varepsilon A_{t,h}^{\underline{\delta}} + (1 - \varepsilon) A_{t,h}^{\delta}$$

for $t, h \geq 0, \varepsilon > 0$

$$\left. \frac{\partial g_{t,h}^{\delta_\varepsilon}}{\partial \varepsilon} \right|_{\varepsilon=0} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left(g_{t,h}^{\delta_\varepsilon} - g_{t,h}^{\underline{\delta}} \right)$$

Definition 11 S^δ **growth optimal** if

$$\left. \frac{\partial g_{t,h}^{\delta_\varepsilon}}{\partial \varepsilon} \right|_{\varepsilon=0} \leq 0$$

for all $t, h \geq 0$ and nonnegative S^δ .

- alternative definition to **expected log-utility**

Kelly (1956)

Hakansson (1971)

Merton (1973a)

Roll (1973)

Markowitz (1976)

Theorem 12 *The numeraire portfolio is growth optimal.*

Strong Arbitrage

- market participants can only exploit arbitrage
- **limited liability**

\implies nonnegative total wealth of each market participant

Definition 13 *A nonnegative S^δ is a **strong arbitrage** if $S_0^\delta = 0$ and*

$$P(S_t^\delta > 0) > 0.$$

Pl. (2002)-mathematical arguments

Loewenstein & Willard (2000)-economic arguments

Theorem 14 *There is **no strong arbitrage**.*

⇒ there is no pricing based on strong arbitrage

- Delbaen & Schachermayer (1998)

free lunches with vanishing risk (FLVR) may exist

- Loewenstein & Willard (2000)

free snacks & cheap thrills may exist

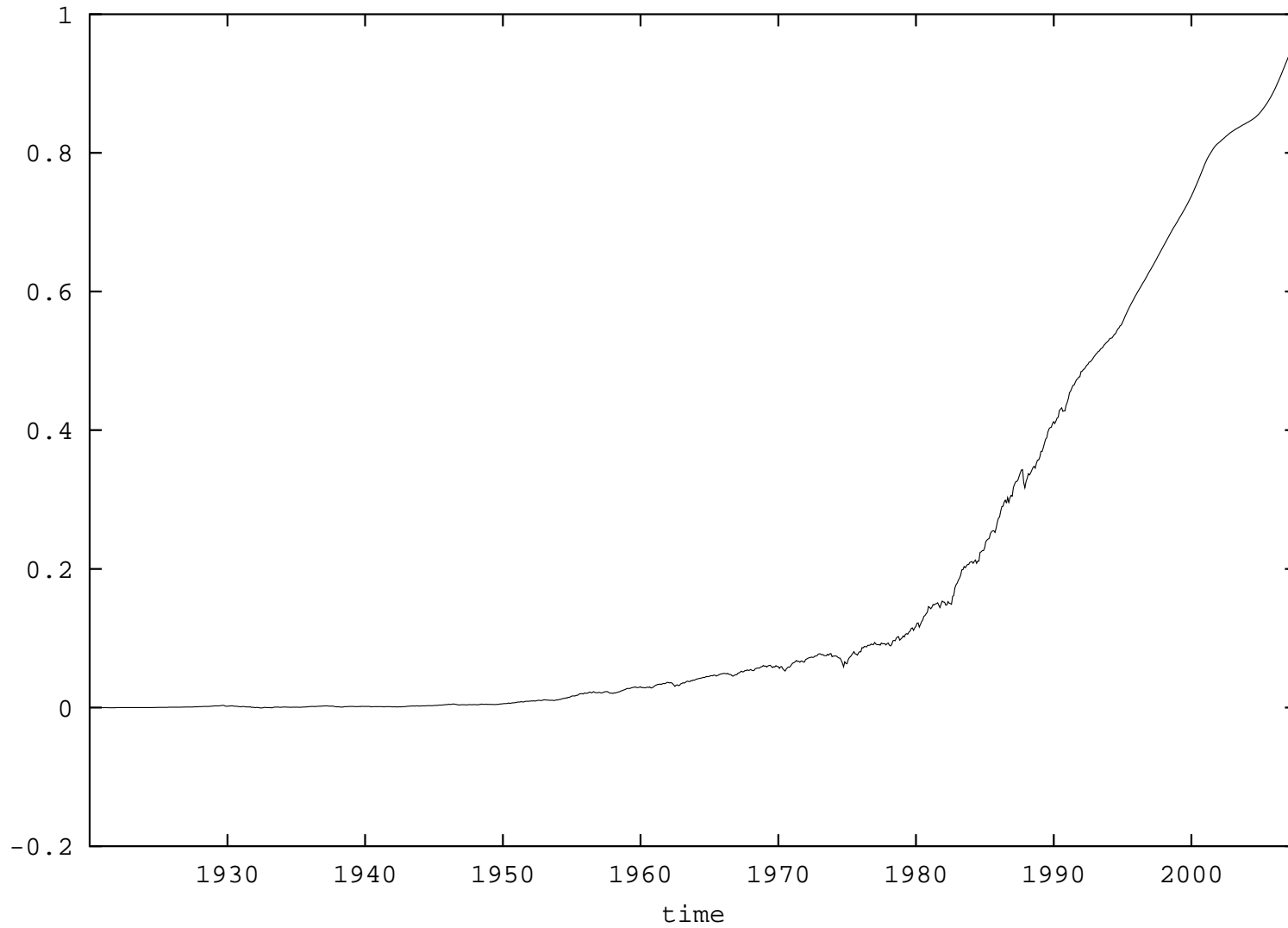


Figure 15: $P(t, T)$ minus savings account.

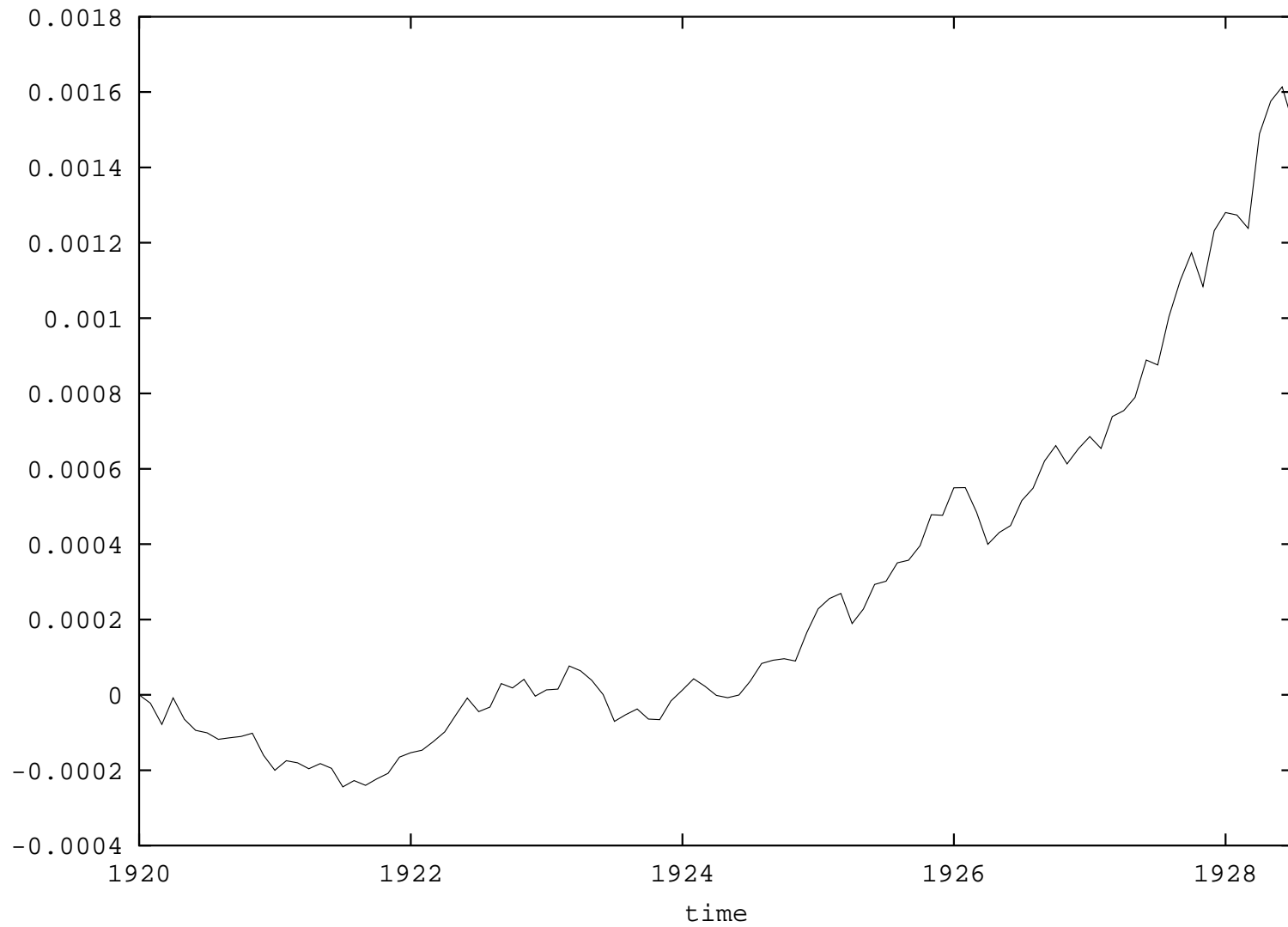


Figure 16: $P(t, T)$ minus savings account.

- Under BA candidate risk neutral Q may *not* be equivalent to P since its Radon-Nikodym derivative may be a strict supermartingale.
- Under BA existence of equivalent risk neutral probability measure is more a **mathematical convenience** than an economic necessity.

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