Generalized Swap Market Model
And the Valuation of Interest Rate Derivatives

Jianwei Zhu
Lucht Probst Associates [1]

November 9, 2007

[1]The corresponding address of the author is: Grosse Gallusstrasse 9, 60311 Frankfurt am Main, Germany. Email is jianwei.zhu@l-p-a.com.

Abstract

In this paper we will establish a generalized Swap Market Model (GSMM) by unifying the stochastic process of swap rates with constant tenors under a single swap measure. GSMM is a natural extension of Libor Market Model (LMM) for swap rates, and LMM can be considered as a special case of GSMM since Libors is a special swap rate with the constant tenor of one period. GSMM can be applied for pricing and hedging any interest rate derivatives, and is suited especially for CMS and swap rate products. There are a number advantages of GSMM: (1). GSMM models swap rates directly, and therefore achieves the best match between products and model. (2). GSMM can be calibrated to the term structure of swaption volatilities easily and quickly. (3) There is no translation of risk sensitivities with respect to swap rates within GSMM. In contrast, risk sensitives such as Vega for swap rates can not be derived directly, and must be translated in an inefficient, inaccurate and nontransparent manner in the most existing interest rate models. (4) All smile modelings for LMM can be taken over for GSMM since GSMM and LMM share an almost identical mathematical structure. (5) GSMM avoids the inconsistency of the market conventions in cap and swaptions markets. Accompanied by these favorite features, GSMM should be a promising interest rate model for pricing and hedging most traded swap rate structures in financial market.

Keywords: Swap Market Model, Libor Market Model, Numeraire change, Swaptions, CMS.

1 Introduction

Both caps and swaptions are the most liquid interest rate derivatives in financial markets. While caps have forward Libors as underlying asset, swaptions are associated with forward swap rates. Swap rates are more complicated constructed than Libors and involve interest rates in more than one periods. In interest rate derivative markets, the prices for caps and swaptions in term of implied volatility are quoted in different manner. This indicates that caps and swaptions are traded in different market segments. However, one of standard pricing models for exotic interest rate structure, most with swap rates as underlying, is Libor Market Model (LMM) (see Brace, Gaterek and Musiela (1997)), and there is no general applicable pricing model based on swap rates. The dynamics of the stochastic processes of Libors in LMM are modeled under so-called forward measures. Hence, LMM is essentially a model for the products with Libors as underlying assets. There are rich and fruitful researches on the extension and application of LMM in the valuation of interest rate derivatives. In order to price swap rate derivatives, LMM must be calibrated to swaption market data. Since swaptions have different swap tenors, the implied swaption volatilities are no longer a volatility surface, but a volatility cube. It is still a great challenge to achieve a good calibration of LMM to swaption volatility cube, even in absence of smile. This calibration problem encountered in LMM is due to the fact that one has to enforce the dynamics and correlations of Libors to match the dynamics of different swap rates. Additionally, an appreciate derivation of risk sensitivities with respect swap rate in LMM is difficult to achieve.

In this paper, we will propose a generalized Swap Market Model (GSMM) to overcome this translation problem. In GSMM, we use a family of swap rates with constant tenors as the modeling objects and build up a consistent and comprehensive model framework for these swap rates under a single swap measure. GSMM is similar to LMM and could be considered as a counterpart to LMM in swap rate world. We do not model all swap rates but a particular subset of forward swap rates, namely, swap rates with constant tenor, for example, the forward swap rates with 5 years swap tenor which are the underlying assets for the $L \times 5Y$ swaptions with L as option length. Forward swap rates with constant tenor are a natural extension of forward Libors to multi-periods. If the swap tenor is one period, for example, 6 months for semi-annual time structure, the swap rates are reduced to 6M Libors. Hence, GSMM is very similar to LMM: While LMM specifies the Libor processes under forward measures associated to zero bond prices, GSMM models the processes of swap rates with constant tenor under swap measure associated with swap annuity. In a standard Swap Market Model (SMM)(see Jamshidian (1997)), each swap rate lives in its own world and follows a driftless geometric Brownian motion under its own swap measure. In this sense, SMM is a model that only valid for a single swap rate, and can not be applied for pricing most exotic interest rate products. It justifies only the use of Black76 formula for swaptions in market. This is the reason why SMM is not extensively discussed in financial literature and not established to be a pricing model. GSMM is a first attempt, to our best knowledge, to unify the separate worlds of SMM for some swap rates under an unique swap measure, and allows us to simulate the underlying swap rate processes consistently, to retrieve other interest rates and zero-coupon bond prices, and then to price and hedge exotic interest rate structure.

SMM does not draw so much attention as LMM both in theoretical side and in industrial side. However recently, Galluccio, Ly, Huang and Scaillet (GLHS)(2007) discussed a general specification of market models in association with swap rates, and extended Jamishidian's formulation by introducing the concept admissibility which characterizes a unique specification of forward swap rate processes in respective swap measure. GLHS particularly considered three sets of swap rates: co-terminal, co-initial and co-sliding swap rates, and discussed co-terminal SMM extensively only. Unfortunately, they built up a SMM without bringing the dynamics of the underlying swap rates together under a single swap measure. In this respect, it is unclear in GLHS's SMM that under which unique swap measure we should simulate swap rates and price interest rate derivatives, which numeraire change we should carry out if we price a product. As we know from LMM, deriving all swap rate processes under a single swap measure such as spot measure is crucial for a complete modeling. In GLHS's notation, we here consider a co-sliding SMM, and will show the numeraire change within such a SMM is very simple and displays an exact functional form as in LMM. Although GSMM is not admissible in the sense that the number of zero-coupon bonds is larger than the number of the available swap rates at each time, we can still derive a yield curve at any future time from the given set of swap rates in a numerical stable way. From the point of view of practical application, bootstrapping a yield (swap) curve from a set of finite swap rates is a job that is done every day in financial institutions.

There are a number of potential advantages of GSMM. (1). Since we directly model the dynamics of particular swap rates, the volatility term structure of swap rates can be easily fitted without extensive calibration. In LMM, the volatility term structure of Libors are often adapted to swaption volatility if a swap rate structure is needed to price. In contrast to LMM, this calibration is not necessary in GSMM. (2). Note that many interest rate derivatives do not have Libors, but swap rates as underlying, for example, CMS structures, it is not always reasonable to work with LMM. Simulating Libors and then computing swap rates is more timing-consuming than directly simulating swap rates. GSMM will be more efficient than LMM in this pricing context. (3). It is much more easily to deal with swaption smile within GSMM . All smile models applied to LMM could be applied to GSMM. Since swap rates are the direct modeling objects, smile volatilities of swaptions are also directly modeled in GSMM. On the other hand, capturing swaption smile effect in LMM is still a great challenge due to its indirect modeling of swaption volatilities. (4). We can set up GSMM in very flexible way in accordance with product underlying. If swap rates with 10 year tenor or 10Y CMS rates play an dominate roll in a product, we should build up a GSMM for swap rates with 10 year tenor. LMM becomes redundant for such products. In a pricing and hedging environment of GSMM, GSMM is set up individually and tailored to individual product specifications. (5). In association with individual model setup for exotic swap rate structures, we can obtain more reasonable and reliable sensitives for hedging since these sensitives are derived from these underlying swap rates (swaptions) which are crucial for hedging the corresponding exotic structure. But in LMM, the sensitives are translated or linked to Libors (caplets), and therefore are not plausible and attainable for hedging decisions. (6). A cumbersome issue of LMM in the valuation of swap rate structures is the different market conventions. As mentioned above, caps and swaptions are traded in the different market segments, and govern by the different market conventions. While liquid caps have mostly quarter or semi-annual periods, the underlying swaps in a swaption is structured annually. Additionally, caps and swaptions have different day counter conventions. This issue is not a modeling issue, but makes the application of LMM to swaptions and swap rate products more difficult, and requires more ad-hoc adjustments. GSMM avoids this market inconsistency.

2 The Model Setup

At first, we define a tenor structure $(T_0 = 0, T_1, T_2, \cdots, T_M)$ on which GSMM are based. Before we begin to set up GSMM, we remind the standard SMM. SMM assumes that a swap rate $S(t, T_n, T_m) = S_{n,m}(t)$ follows a driftless geometric Brownian motion under the corresponding swap measure Q^n

$$dS_{n,m}(t) = S_{n,m}(t)\sigma dW^n(t), \tag{1}$$

where the swap rate $S_{n,m}(t)$ is defined by

$$S_{n,m}(t) = \frac{B_n(t) - B_m(t)}{\sum_{h=n+1}^{m} \tau_h B_h(t)} = \frac{B_n(t) - B_m(t)}{N_{n,m}(t)}, \tag{2}$$

$B_i(t)$ denotes time t price for a zero coupon bond maturing at time T_i. If $t > T_n$, the swap rate $S_{n,m}(t)$ is not defined, or in other words, dies. τ_i is the year fraction for time interval $[T_{i-1}, T_i]$. $N_{n,m}(t) = \sum_{h=n+1}^{m} \tau_h B_h(t)$ is referred to swap annuity and serves as the numeraire for the corresponding swap measure Q^n. In this particular SMM for swap rate $S_{n,m}(t)$, we can derive the industrial standard pricing formula for swaption, the so-called

Black'76 formula. A payer swaption at time T_n has the following payoff

$$\begin{aligned} SW_{Payer}(T_n) &= \big[(B_n(T_n) - B_m(T_n)) - K \sum_{h=n+1}^{m} \tau_h B_h(T_n)\big]^+ \\ &= [S_{n,m}(T_n)N_{n,m}(T_n) - KN_{n,m}(T_n)]^+. \end{aligned}$$

Denote $M(T_n) = e^{\int_0^{T_n} r(u)du}$ as the money market account up to T_n, which is the corresponding numeraire for the risk-neutral measure Q. We define now a Radon-Nikodym derivative

$$\frac{dQ}{dQ^n} = \frac{M(t)N_{n,m}(0)}{M(0)N_{n,m}(t)}$$

with $M(0) = 1$. According to the risk-neutral pricing a payer swaption can be valued as follows

$$\begin{aligned} SW_{Payer}(0) &= \mathbf{E}^Q\big[\frac{1}{M(T_n)}[S_{n,m}(T_n)N_{n,m}(T_n) - KN_{n,m}(T_n)]^+\big] \\ &= \mathbf{E}^Q\big[\frac{N_{n,m}(T_n)}{M(T_n)}[S_{n,m}(T_n) - K]^+\big] \\ &= \mathbf{E}^{Q^n}\big[\frac{N_{n,m}(T_n)}{M(T_n)}\frac{M(T_n)N_{n,m}(0)}{N_{n,m}(T_n)}[S_{n,m}(T_n) - K]^+\big] \\ &= N_{n,m}(0)\mathbf{E}^{Q^n}\big[[S_{n,m}(T_n) - K]^+\big], \end{aligned}$$

where the risk-neutral measure Q is changed to the swap measure Q^n. As the swap rate $S_{n,m}(t)$ is specified to be a driftless geometric Brownian motion under the measure Q^n in SMM, it is straightforward to obtain the swap pricing formula

$$SW_{Payer}(0) = N_{n,m}(0)\big[S_{n,m}(T_n)\Phi(d_+) - K\Phi(d_-)\big] \tag{3}$$

with

$$d_{\pm} = \frac{\ln S_{n,m}(0)/K \pm \frac{1}{2}\sigma^2 T_n}{\sigma\sqrt{T_n}}.$$

Here $\Phi(\cdot)$ denotes the cumulative standard normal distribution. Similarly to Black-Scholes equity option pricing formula, there is also a one to one correspondence between swaption price and its volatility. The implied volatilities from the above Black'76 formula are quoted in market and display a smile/skew pattern. The essence of SMM for the swap rate $S_{n,m}(t)$ is that $S_{n,m}(t)$ is a martingale under the swap measure Q^n associated with the numeraire $N_{n,m}(t)$. However, this swap measure Q^n is only valid for the swap rate $S_{n,m}(t)$, and can not be applied for other swap rates without any change of measures. Since we can construct a large number of swap rates by varying the index n and m, and each swap rate is living with its own swap measure, it is impossible to bring all of them into an unique world. SMM is then a model for a single rate and becomes useless for the valuation of exotic interest rate derivatives.

In this paper, we propose a Generalized Swap Market Model (GSMM) and try to fill the gap of SMM by unifying the swap rates with constant tenor under the single swap measure. As shown below, the numeraire changes for this set of swap rates is simple, and much easier than for other sets of swap rates, for example, co-terminal swap rates. We consider a set of forward swap rates with constant tenor $\{S_i^c(t)\}_{0<i\le M-c}$,

$$S_{i,i+c}(t) = S_i(t) = \frac{B_i(t) - B_{i+c}(t)}{\sum_{h=i+1}^{i+c} \tau_h B_h(t)} = \frac{B_i(t) - B_{i+c}(t)}{N_i(t)} \tag{4}$$

with

$$N_{i,i+c}(t) = N_i(t) = \sum_{h=i+1}^{i+c} \tau_h B_h(t).$$

The positive integer c is the tenor length of the swap rates we are considering. In the following, we will suppress c in $S_{i,i+c}(t)$ for notational simplicity. This set of swap rates $\{S_i(t)\}_{0<i\le M-c}$ are the modeling rates in GSMM. It is easy to see the similarity between Libors and $S_i(t)$. For $c = 1$, $S_i(t)$ is reduced to a Libor,

$$L_i(t) = S_i(t) = \frac{B_i(t) - B_{i+1}(t)}{\tau_i B_{i+1}(t)} = (B_i(t)/B_{i+1}(t) - 1)/\tau_i.$$

At the same time, swap annuity $N_i(t) = \tau_{i+1} B_{i+1}(t)$ degenerates a τ_{i+1} fraction of zero-bond price $B_{i+1}(t)$, which serves as the numeraire for forward measure in LMM.[1] LMM could be regarded as a special case of GSMM. $\{S_i(t)\}_{0<i\le M-c}$ is a natural set of swap rates with which we can build up a comprehensive swap rate model.

Under the numeraire N_i, $S_i(t)$ are assumed to follow a driftless geometric Brownian motion as in SMM

$$dS_i(t) = S_i(t)\sigma_i dW_i(t).$$

In GSMM we take the correlation among $\{S_i(t)\}_{0<i\le M-c}$ into account, and assume $dW_i dW_j = \rho_{ij} dt$.

We now fix a special swap numeraire $N_k(t)$, and derive all processes of $\{S_i(t)\}_{0<i\le M-c}$ under $N_k(t)$. Denote Q^i for the swap measure associated with $N_i(t)$, we define a measure change as follows

$$\frac{dQ^i}{dQ^{i+1}} = \frac{N_i(t) N_{i+1}(0)}{N_{i+1}(t) N_i(0)}. \tag{5}$$

[1] The fraction τ_{i+1} has only a scaling effect on the corresponding measure.

Note

$$N_i(t) - N_{i+1}(t) = \tau_{i+1}B_{i+1}(t) - \tau_{i+c+1}B_{i+c+1}(t) = \tau[B_{i+1}(t) - B_{i+c+1}(t)]$$

where it is assumed that the year fractions τ_{i+1} and τ_{i+c+1} are equal to τ.[2] We obtain

$$\begin{aligned}\frac{N_i(t)}{N_{i+1}(t)} &= \frac{S_{i+1}(t)[B_i(t) - B_{i+c}(t)]}{S_i(t)[B_{i+1}(t) - B_{i+c+1}(t)]} \\ &= \frac{S_{i+1}(t)\tau^{-1}[N_{i-1}(t) - N_i(t)]}{S_i(t)\tau^{-1}[N_i(t) - N_{i+1}(t)]} \\ &= \frac{S_{i+1}(t)[N_{i-1}(t) - N_i(t)]}{S_i(t)[N_i(t) - N_{i+1}(t)]}. \qquad (6)\end{aligned}$$

Now we propose the following function form for $\frac{N_i(t)}{N_{i+1}(t)}$

$$\frac{N_i(t)}{N_{i+1}(t)} = x(t) = 1 + \alpha S_{i+1}(t). \qquad (7)$$

and show it is an exact solution for this numeraire change. According to this function form we have

$$N_i(t) = N_{i+1}(t) + \alpha S_{i+1}(t)N_{i+1}(t)$$

or

$$N_i(t) - N_{i+1}(t) = \alpha S_{i+1}(t)N_{i+1}(t).$$

Inserting this function into (6) yields

$$\begin{aligned}\frac{N_i(t)}{N_{i+1}(t)} &= \frac{S_{i+1}(t)\alpha S_i(t)N_i(t)}{S_i(t)\alpha S_{i+1}(t)N_{i+1}(t)} \\ &\equiv \frac{N_i(t)}{N_{i+1}(t)}. \qquad (8)\end{aligned}$$

This shows that $\frac{N_i(t)}{N_{i+1}(t)}$ is exactly equal to $1 + \alpha S_{i+1}(t)$. To obtain α, we set $t = 0$

$$\frac{N_i(0)}{N_{i+1}(0)} = 1 + \alpha S_{i+1}(0)$$

This leads to

$$\alpha = \Big(\frac{N_i(0)}{N_{i+1}(0)} - 1\Big)/S_{i+1}(0) = \tau.$$

The Radon-Nikodym derivative now can be rewritten as

$$\frac{dQ^i}{dQ^{i+1}} = \frac{1 + \tau S_{i+1}(t)}{1 + \tau S_{i+1}(0)}. \qquad (9)$$

[2]This assumption is well justified because the day counter convention for standard swap in financial market is 30/360. According to this convention, all year fractions τ_i are equal.

This seems very similar to the Radon-Nikodym derivative used in LMM, which is

$$\frac{dP^{i+1}}{dP^{i+2}} = \frac{1 + \tau L_{i+1}(t)}{1 + \tau L_{i+1}(0)}.$$

with P^j as forward measure associated with $B_j(t)$.

By applying Girsanov theorem, we can derive the process of $S_i(t)$ under the swap measure Q^{i+1}. Repeating this numeraire change step by step yields the process of $S_i(t)$ under the swap measure $Q^k, i < k$. The technique applied here is the same one as in LMM (see Musiela and Rutkowski (2004) or Zhu (2007)). Generally, the processes of swap rates $\{S_i(t)\}_{0<i\leq M-c}$ under an arbitrary swap measure $N_k(t)$ take the following form

$$\frac{dS_i(t)}{S_i(t)} = \mu_i(t)dt + \sigma_i(t)dW_i(t), \tag{10}$$

with

$$\mu_i = \sigma_i(t) \sum_{h=k+1}^{i} \frac{\sigma_h(t)\rho_{hi}\tau S_h(t)}{1 + \tau S_h(t)}, \quad i > k;$$

$$\mu_i = 0, \quad i = k;$$

$$\mu_i = -\sigma_i(t) \sum_{h=i+1}^{k} \frac{\sigma_h(t)\rho_{hi}\tau S_h(t)}{1 + \tau S_h(t)}, \quad i < k.$$

If $k = 0$, we call the swap measure $N_0(t) = \sum_{h=1}^{c} \tau_h B_h(t)$ as spot swap measure, that is the counterpart of the spot measure in LMM. Similarly, by setting $k = M - c$ we obtain the latest measure for swap rates $\{S_i(t)\}_{0<i\leq M-c}$, that could be referred to as terminal swap measure. Both measures play a central roll in the practical simulation of GSMM.

Deriving the stochastic processes for $\{S_i(t)\}_{0<i\leq M-c}$ under a single swap measure is the first step of establishing a complete GSMM. A complete interest rate model should be able to deliver zero-bond prices at each future time. Particularly for GSMM, we should be able to derive the zero-bond prices at times $(T_1, T_2, \cdots, T_M)$. To this end, we consider the swap rates at time T_j, namely $(\tilde{S}_j(T_j), \tilde{S}_{j+1}(T_j), \cdots, \tilde{S}_{M-c}(T_j))$. The symbol ˜ emphasizes that the swap rates evolute up to time T_j stochastically. Using these swap rates we try retrieve the zero bond prices $(\tilde{B}_{j+1}(T_j), \tilde{B}_{j+2}(T_j), \cdots, \tilde{B}_M(T_j))$. In other words, given $(M-c-j+1)$ swaps at time T_j, we have to bootstrap $M-j$ zero-bond prices at that time point. Bootstrapping a yield curve from a set of swap rates is a day-to-day business in every bank.

There is a simple and efficient way to retrieve the time T_j yield curve. We start with the last swap rate $\tilde{S}_{M-c}(T_j)$,

$$\tilde{S}_{M-c}(T_j) = \frac{\tilde{B}_{M-c}(T_j) - \tilde{B}_M(T_j)}{\sum_{h=M-c+1}^{M} \tau_h \tilde{B}_h(T_j)}.$$

where $c+1$ unknown prices $\tilde{B}_h(T_j), M-c \leq h \leq M$, have to be solved for. Note all prices of zero coupon bonds appearing in $\tilde{S}_{M-c}(T_j)$ lie in the last end of a yield curve, and will converge to a stable level. Hence we could suggest a functional form for $\tilde{B}_h(T_j)$ as follows

$$\tilde{B}_h(T_j) = \beta_0 e^{-\beta_1(T_h - T_j)} \tag{11}$$

with β_0 and β_1 as two constants. This functional form coincides with the zero-bond prices implied in most term structure models (Vasicek(1977), Cox, Ingersoll and Ross (1985)). Plugging this zero-bond price form in $\tilde{S}_{M-c}(T_j)$ yields

$$\tilde{S}_{M-c}(T_j) = \frac{e^{-\beta_1(T_{M-c}-T_j)} - e^{-\beta_1(T_M - T_j)}}{\sum_{h=M-c+1}^{M} \tau_h e^{-\beta_1(T_h - T_j)}}, \tag{12}$$

where the parameter β_0 has vanished. We can solve the equation (12) for the parameter β_1 quickly by using simple numerical routine. Additionally, we note that the swap rate $\tilde{S}_{M-c}(T_j)$ could be interpreted as an average interest rate for the time period $[T_{M-c}, T_M]$ at time T_j. This interpretation leads to the following relation

$$\exp\big(-\tilde{S}_{M-c}(T_j)[\frac{1}{2}(T_M + T_{M-c}) - T_j]\big) = \beta_0 \exp\big(-\beta_1[\frac{1}{2}(T_M + T_{M-c}) - T_j]\big) \tag{13}$$

which results in

$$\beta_0 = \exp\big(-[\tilde{S}_{M-c}(T_j) - \beta_1][\frac{1}{2}(T_M + T_{M-c}) - T_j]\big). \tag{14}$$

Given β_0 and β_1, we are able to determine the zero-bond prices ($\tilde{B}_{M-c}(T_j)$, $\tilde{B}_{M-c+1}(T_j), \cdots, \tilde{B}_M(T_j)$), and then try to retrieve $\tilde{B}_{M-c-1}(T_j)$ by using the swap rate $\tilde{S}_{M-c-1}(T_j)$,

$$\tilde{S}_{M-c-1}(T_j) = \frac{\tilde{B}_{M-c-1}(T_j) - \tilde{B}_{M-1}(T_j)}{\sum_{h=M-c}^{M-1} \tau_h \tilde{B}_h(T_j)}.$$

By rearranging the above equation we obtain directly $\tilde{B}_{M-c-1}(T_j)$

$$\tilde{B}_{M-c-1}(T_j) = \tilde{S}_{M-c-1}(T_j) \sum_{h=M-c}^{M-1} \tau_h \tilde{B}_h(T_j) + \tilde{B}_{M-1}(T_j), \tag{15}$$

where all terms on the right hand of the above equation are known. Repeating this procedure results in a bootstrapped yield curve ($\tilde{B}_{j+1}(T_j)$, $\tilde{B}_{j+2}(T_j)$, $\cdots$, $\tilde{B}_M(T_j)$). Based on this yield curve we can construct any other swap rates as well as Libors.

3 Valuation and Simulation

Let $G(T_j)$ denote the payoff function at time T_j of an interest rate derivative. The discounted payoff of $G(T_j)$ at time $T_p, p < j$, under risk-neutral measure is

$$\frac{G(T_j)}{M(T_p, T_j)} = \frac{G(T_j)M(T_p)}{M(T_j)}.$$

The value of this derivative at time T_p under the swap measure Q^k reads

$$G(T_p) = M(T_p)\mathbf{E}^Q\big[\frac{G(T_j)}{M(T_j)}\big] = N_k(T_p)\mathbf{E}^{Q^k}\big[\frac{G(T_j)}{N_k(T_j)}\big] \tag{16}$$

where we use the Radon-Nikodym derivative $\frac{dQ}{dQ^k}$ again to allow us price G under the given swap measure. This pricing equation can be used to price path-dependent interest rate structures. Particularly we have the present value of $G(T_j)$ at time 0

$$G(0) = N_k(0)\mathbf{E}^{Q^k}\big[\frac{G(T_j)}{N_k(T_j)}\big] \tag{17}$$

A standard swaption with payoff $G(T_n) = N_n(T_n)[S_n(T_n)-K]^+$ is valued as

$$SW_{Payer}(0) = N_k(0)\mathbf{E}^{Q^k}\big[\frac{N_n(T_n)[S_n(T_n) - K]^+}{N_k(T_n)}\big]$$

which is simplified for $n = k$ to

$$SW_{Payer}(0) = N_n(0)\mathbf{E}^{Q^n}[[S_n(T_n) - K]^+].$$

This is equivalent to Black76 formula.

In GSMM the instantaneous correlation of two different Brownian motions W_i and W_j is given by ρ_{ij}, which generally does not affect the swaption prices. However, the correlation is crucial for pricing some exotic products. for example, CMS memory and CMS spreads. CMS memory is a product paying variable coupons that are dependent the current CMS rate and the previous CMS rates. Therefore, this product is specially sensitive to the correlations between the CMS rates. GSMM is suited very well to this memory product since GSMM models the relevant CMS rates (swap rates) directly, and the instantaneous correlations in GSMM also describe the correlation structure of these CMS rates directly. For the pricing CMS spread products that involve two different types of CMS rates, for example, CMS10-CMS2, we can choose one type of CMS rate as underlying modeling object and left the other type of CMS rate to be calibrated by fitting correlations. In this sense, GSMM has one dimension less to consider than LMM: GSMM does not need to calibrate two different type CMS rates simultaneously as in LMM, but to calibrate only one type CMS rate. Therefore, GSMM will perform much faster and more efficiently than LMM, and we win a number of benefits for CMS products which make up a large portion of the traded interest rate derivatives in financial market. The correlation matrix of the Brownian motions $W_i, 1 \leq i \leq M-c$, could be parameterized according to the approaches suggested by Schoenmackers and Coffey (2000), or Rebonato (1998). The parsimonious parameterization makes the calibration more easy.

For a practical application of GSMM, we need to simulate GSMM given in equation (10) under the swap measure Q^k. The Euler discretization of equation (10) is

$$S_i(T_{h+1}) = S_i(T_h) + S_i(T_h)[\mu_i(T_h)\tau_{h+1} + \sigma_i(T_h)Z_i(T_h)\tau_{h+1}], \tag{18}$$

where $Z_i(T_h), h \leq i \leq M-c$, are uncorrelated standard normal distributed random variables, which are drawn at time T_h. The simulation is straightforward and similar to the simulation of LMM. A more sophisticated simulation is based on the process of $\ln S_i(t)$ and is given by

$$S_i(T_{h+1}) = S_i(T_h)\exp[(\mu_i(T_h) - \frac{1}{2}\sigma_i^2(T_h))\tau_{h+1} + \sigma_i(T_h)Z_i(T_h)\tau_{h+1}]. \tag{19}$$

Combined with Brownian bridge simulation, GSMM can be applied easily for pricing CMS range accruals. CMS range accruals is an interest rate derivatives paying coupons depending upon the number of days in which underlying CMS rate moves within a given range. In contrast, it is difficult to obtain a satisfactory simulation for such a product in LMM.

The volatility $\sigma_i(t)$ is the instantaneous volatility, and is also referred to term volatility in a discrete setup, which is generally not identical to the implied volatility $\bar{\sigma}_i$ that is retrieved from Blaack76 formula. For the time being, we assume that the volatility $\sigma_i(t)$ is piecewise constant over the given time partition, and neglect the smile effect of $\sigma_i(t)$. There is the following relation between $\sigma_i(t)$ and $\bar{\sigma}_i$

$$\bar{\sigma}_i^2 T_i = \int_0^{T_i} \sigma_i^2(t)dt = \sum_{h=1}^{i} \sigma_h^2(T_{h-1})\tau_h.$$

Given the implied volatilities $\{\bar{\sigma}_i\}_{1\leq i \leq M-c}$, we can derive the piecewise constant volatilities $\sigma_(t)$ and apply them to simulate the simple GSMM. Parameterized term structure of volatility can also be used to fit the implied volatilities. A good and widely applied formulation for term structure of volatility is the following parameterization with four parameters a_0, a_1, a_2, a_3,

$$\sigma(T_i) = a_0 + (a_1 T_i + a_2)e^{a_3 T_i}, \tag{20}$$

which allows for humped, increasing and decreasing term structure.

4 Smile Modeling within GSMM

Generally all smile models extending LMM can be taken over to set up smile models within GSMM. The models regarding smile modeling mainly fall into two categories: local volatility models and stochastic volatility models. To illustrate how to establish a smile GSMM analogy to the smile LMM, we list some possible extensions for SMM:

1. Local Volatility Model á la Andersen and Andreasen (1998). $S_i(t)$ follows a displaced constant elasticity of variance (CEV) process under the respective swap measure Q^i

$$dS_i(t) = [S_i^b(t) - a]\sigma_i(t)dW_i(t), \qquad a > 0, 0 \leq b \leq 1. \tag{21}$$

that will becomes under spot swap measure Q^1

$$dS_i(t) = [S_i^b(t) - a][\mu_i(t) + \sigma_i(t)]dW_i^{Q^1}(t), \qquad a > 0, 0 \leq b \leq 1. \tag{22}$$

with

$$\mu_i(t) = \sum_{j=1}^{i} \frac{\tau\rho_{ij}\sigma_j\sigma_i[S_i^b(t) - a]}{1 + \tau S_i(t)}.$$

Since both displacement and CEV produce only skew effect, this smile modeling of GSMM does not allow symmetric smile pattern.

2. Stochastic Volatility Model á la Andersen and Brotherton-Raticiliff (2001). In this smile formulation, $S_i(t)$ has a mean-reverting squared root process $V(t)$ as a part of variance under the respective swap measure Q^i,

$$\begin{aligned} dS_i(t) &= \phi(S_i(t))\sigma_i(t)\sqrt{V(t)}dW_i(t), \\ dV_(t) &= \kappa(\theta - V(t))dt + \nu\sqrt{V(t)}dZ(t), \end{aligned} \tag{23}$$

where $W_i(t)$ and $Z(t)$ are uncorrelated. Due to zero correlation between swap rates and its stochastic volatility component we need additional mechanism to generate a skew effect.

3. Local/Stochastic Volatility Model á la Piterbarg (2003). This smile modeling is a mixture of local volatility and stochastic volatility. $S_i(t)$ follows the following process under the respective swap measure Q^i,

$$\begin{aligned} dS_i(t) &= [b_iS_i(t) + (1 - b_i)S_i(0)]\sigma_i(t)\sqrt{V(t)}dW_i(t), \\ dV_(t) &= \kappa(\theta - V(t))dt + \nu\sqrt{V(t)}dZ(t), \end{aligned} \tag{24}$$

where $W_i(t)$ and $Z(t)$ are uncorrelated. This model can also be regarded as a special case of Andersen and Brotherton-Raticiliff model, and introduces displacement to generate volatility skew.

4. Stochastic Volatility Model á la Wu and Zhang (2002). This smile model takes the following form under the respective swap measure Q^i,

$$\begin{aligned} dS_i(t) &= S_i(t)\sigma_i(t)\sqrt{V(t)}dW_i(t), \\ dV(t) &= \kappa(\theta - V(t) - \xi_i(t))dt + \nu\sqrt{V(t)}dZ(t), \end{aligned} \tag{25}$$

where $\xi_i(t$ is a drift adjustment due to the change of risk-neutral measure to measure Q^i. $W(t)$ and $Z(t)$ are correlated,

$$\frac{< \sigma_i(t) dW_i(t) > dZ(t)}{|\sigma_i(t)|} = \rho_i dt.$$

All swap rates share a single volatility process.

5. Stochastic Volatility Model á la Zhu (2007). Zhu proposes a smile modeling for LMM with individual stochastic volatility process and individual volatility correlation. This model can be applied for $S_i(t)$ under respective swap measure Q^i,

$$\begin{aligned} dS_i(t) &= S_i(t)\sqrt{V_i(t)}dW_i(t), \\ dV_i(t) &= \kappa_i(\theta_i - V(t))dt + \nu_i\sqrt{V_i(t)}dZ(t), \end{aligned} \tag{26}$$

with $dW_i(t)dZ(t) = \rho_i$. The parameters κ_i, θ_i, ν_i and $V_i(0)$ are parameterized by a nesting function respectively, which allow for a parsimonious formulation of this smile model.

Needless to say, we should carry out the corresponding numeraire changes for all above stochastic volatility models in order to obtain an unified world for swap rates and stochastic volatilities. The above five examples for possible extensions of GSMM show that GSMM can be not only embedded easily into the existing smile models for Libors , but also share the same implementations and software architecture as for LMM. In this sense, we do not discuss this issue extensively.

5 Conclusions

In this paper we established a generalized Swap Market Model and unified the dynamics of the swap rates with constant tenors in a single swap measure. Using these swap rates we can retrieve zero bond prices and obtain a yield curve at every time. Hence we can price any interest rate derivatives under a given swap measure and make swap measure as a pricing measure for general pricing and hedging purposes. GSMM could be considered as counterpart of LMM, or more generally as shown in the paper, it includes LMM as special case since Libors are special swap rates with constant tenor of one period. GSMM is equipped very well for exotic CMS products because the underlying swap rates in such products coincide with the modeling objects in GSMM. This perfect match between products and model allows for plausible and transparent hedging and risk management, and voids the translation of risk sensitivities of swap rates in LMM. However, GSMM is not a replacement for LMM for the Libor sensitive products. GSMM can be extended to model swaption volatility smile/skew by simply taking over the existing smile models for LMM. Accompanied by these favorite features, GSMM should be a promising interest rate model for pricing and hedging most traded swap rate structures in financial market.

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