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Generalized Swap Market Model
And the Valuation of Interest Rate Derivatives

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Contents

1 Motivations

Why we need a new model for swap products? We have a number of reasons for an applicable swap market model.

From the point of view of market:

(i) Although caps and swaptions are both liquid IR derivatives, they are quoted in different ways in terms of implied cap volatilities and swaption volatilities.

(ii) The underlyings of caps and swaptions are Libors and swap rates respectively, which are also govern by different market conventions.

(iii) The arbitrage between cap market and swaption market is very restrictive.

The above facts indicates that cap and swaption markets are two different market segments within IR derivative market.

From the point of view of modeling:

(i) The dominate market standard model for pricing IR derivatives is Libor Market Model (LMM), which is based on the dynamics of of a family of Libors, and the correlation structure between them.

(ii) The calibration of LMM to swap market is to enforce the dynamics of Libors and the correlation structure to fit the (selected) swaption prices. This is not an easy job and fails often.

(iii) There is no efficient, transparent and consistent way to calculate the risky sensitivities for swap rate, for example, Vega, in LMM.

(iv) There is obviously no match between model and products, if we apply LMM to price a swap rate related products.

Why we do not establish a model which focuses on swaption market directly?

2 Swap Market Model (SMM) So Far

Jamshidian (1997) justified the market standard pricing formula Black76 for swaptions by introducing swap annuity (level process, PVBP) as numeraire and the related swap measure. A payer swaption has the following payoff at

time T_n ,

$$\begin{aligned} SW_{Payer}(T_n) &= \left[(B_n(T_n) - B_m(T_n)) - K \sum_{h=n+1}^{m} \tau_h B_h(T_n)\right]^+ \\ &= [S_{n,m}(T_n)N_{n,m}(T_n) - KN_{n,m}(T_n)]^+. \end{aligned}$$

with

$$S_{n,m}(t) = \frac{B_n(t) - B_m(t)}{\sum_{h=n+1}^{m} \tau_h B_h(t)} = \frac{B_n(t) - B_m(t)}{N_{n,m}(t)}.$$

$N_{n,m} = \sum_{h=n+1}^{m} \tau_h B_h(t)$ is referred to as swap annuity.

Denote $M(T_n) = e^{\int_0^{T_n} r(u)du}$ as the money market account up to T_n, which is the corresponding numeraire for the risk-neutral measure Q. Now define a

Radon-Nikodym derivative

$$\frac{dQ}{dQ^n} = \frac{M(t)N_{n,m}(0)}{M(0)N_{n,m}(t)}$$

with $M(0) = 1$.

$$\begin{aligned}
SW_{Payer}(0) &= \mathbf{E}^{Q}\Big[\frac{1}{M(T_n)}[S_{n,m}(T_n)N_{n,m}(T_n) - KN_{n,m}(T_n)]^{+}\Big] \\
&= \mathbf{E}^{Q}\Big[\frac{N_{n,m}(T_n)}{M(T_n)}[S_{n,m}(T_n) - K]^{+}\Big] \\
&= \mathbf{E}^{Q^n}\Big[\frac{N_{n,m}(T_n)}{M(T_n)}\frac{M(T_n)N_{n,m}(0)}{N_{n,m}(T_n)}[S_{n,m}(T_n) - K]^{+}\Big] \\
&= N_{n,m}(0)\mathbf{E}^{Q^n}\Big[[S_{n,m}(T_n) - K]^{+}\Big],
\end{aligned}$$

where the risk-neutral measure Q is changed to the swap measure Q^n. Assume $S_{n,m}(t)$ follows a driftless geometric Brownian motion under the measure Q^n, we can price swaptions as follows,

$$SW_{Payer}(0) = N_{n,m}(0)\left[S_{n,m}(T_n)\Phi(d_+) - K\Phi(d_-)\right] \quad (2.1)$$

with

$$d_\pm = \frac{\ln S_{n,m}(0)/K \pm \frac{1}{2}\sigma^2 T_n}{\sigma\sqrt{T_n}}.$$

Here $\Phi(\cdot)$ denotes the cumulative standard normal distribution.

Some facts about SMM:

(i) Each swap rate has its own swap measure. The different swap rates do not live in an unified world.

(ii) In literature, swap rates are usually classified into three groups: co-terminal, co-initial and co-sliding swap rates. And SMM is then correspondingly classified into the above three groups.

(iii) Co-terminal SMM has drawn the most attention from financial research community. See Jamshidian (1997), Musiela and Rutkowski (2002), Galluccio et al. (2003). Co-terminal SMM is suitable for pricing Bermudian swaptions, but not for CMS products.

(iv) The importance of co-sliding SMM for building up an applicable swap market model is ignored due to the argument of admissibility (Galluccio et al. (2003)).

3 Generalized (Co-Sliding) Swap Market Model

3.1 Co-Sliding vs. Co-Terminal

My purpose here: build up a GSMM parallel to LMM.

(i) Which underlying swap rates would be taken into account in GSMM? Co-sliding swap rates, or swap rates with constant tenor, is a natural extension of Libors. Therefore, co-sliding SMM is our candidate for GSMM, which completes co-sliding SMM.

(ii) What are the troubles with co-terminal SMM? The underlying swap rates in co-terminal SMM have different tenors.

(a). The underlings are not homogenous in terms of swap tenor.

(b). We can not bootstrap a reasonable volatility term structure from the co-terminal swaptions.

(c). The correlation structure of co-terminal swaptions is not plausible.

(iii) In contrast, co-sliding SMM has many favorite features.

(a). For example, we can calculate the forward volatility for time period $[2, 3]$ from 2×5 swaption volatility and 3×5 swaption volatility.

(b). There exists a natural correlation matrix for co-sliding swap rates, as for Libors.

(c). LMM can be considered as a special case of GSMM.

3.2 Numeraire Change

Now we consider a set of forward swap rates with constant tenor $\{S_i^c(t)\}_{0<i\leq M-c}$,

$$S_{i,i+c}(t) = S_i(t) = \frac{B_i(t) - B_{i+c}(t)}{\sum_{h=i+1}^{i+c} \tau_h B_h(t)} = \frac{B_i(t) - B_{i+c}(t)}{N_i(t)} \tag{3.1}$$

with

$$N_{i,i+c}(t) = N_i(t) = \sum_{h=i+1}^{i+c} \tau_h B_h(t).$$

The positive integer c is the tenor length of the swap rates. The swap rate $S_i(t)$ follows a driftless geometric Brownian motion

$$dS_i(t) = S_i(t)\sigma_i dW_i(t). \tag{3.2}$$

under the corresponding swap measure Q^i with the swap annuity N_i.

If $c = 1$, $S_i(t)$ is reduced to be a Libor. GSMM will become a LMM.

As the first step of building up GSMM, we will unify all swap rates $\{S_i(t)\}_{0<i\leq M-c}$ under a single swap measure, for example, N_k. We define a measure change as follows

$$\frac{dQ^i}{dQ^{i+1}} = \frac{N_i(t)N_{i+1}(0)}{N_{i+1}(t)N_i(0)}. \tag{3.3}$$

Note

$$N_i(t) - N_{i+1}(t) = \tau_{i+1}B_{i+1}(t) - \tau_{i+c+1}B_{i+c+1}(t) = \tau[B_{i+1}(t) - B_{i+c+1}(t)]$$

we can verify

$$\frac{N_i(t)}{N_{i+1}(t)} = x(t) = 1 + \tau S_{i+1}(t). \tag{3.4}$$

which implies that the Radon-Nikodym derivative can be rewritten now as

$$\frac{dQ^i}{dQ^{i+1}} = \frac{1 + \tau S_{i+1}(t)}{1 + \tau S_{i+1}(0)}. \tag{3.5}$$

This seems very similar to the Radon-Nikodym derivative used in LMM, which is

$$\frac{dP^{i+1}}{dP^{i+2}} = \frac{1 + \tau L_{i+1}(t)}{1 + \tau L_{i+1}(0)}.$$

with P^j as forward measure associated with $B_j(t)$. Repeating this numeraire change step by step yields the process of $S_i(t)$ under the swap measure $Q^k, i < k$.

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Generally, the processes of swap rates $\{S_i(t)\}_{0<i\leq M-c}$ under a fixed arbitrary swap measure $N_k(t)$ take the following form

$$\frac{dS_i(t)}{S_i(t)} = \mu_i(t)dt + \sigma_i(t)dW_i(t), \tag{3.6}$$

with

$$\mu_i = \sigma_i(t)\sum_{h=k+1}^{i}\frac{\sigma_h(t)\rho_{hi}\tau S_h(t)}{1+\tau S_h(t)}, \quad i>k;$$
$$\mu_i = 0, \quad i=k;$$
$$\mu_i = -\sigma_i(t)\sum_{h=i+1}^{k}\frac{\sigma_h(t)\rho_{hi}\tau S_h(t)}{1+\tau S_h(t)}, \quad i<k.$$

(i) The mathematical structure of $S_i(t)$ under an arbitrary swap measure is

identical to these of Libor processes in LMM. GSMM is a natural extension of LMM.

(ii) If $k = 0$, we call the swap measure $N_0(t) = \sum_{h=1}^{c} \tau_h B_h(t)$ as spot swap measure, that is the counterpart of the spot measure in LMM. Similarly, by setting $k = M - c$ we obtain the latest measure as terminal swap measure.

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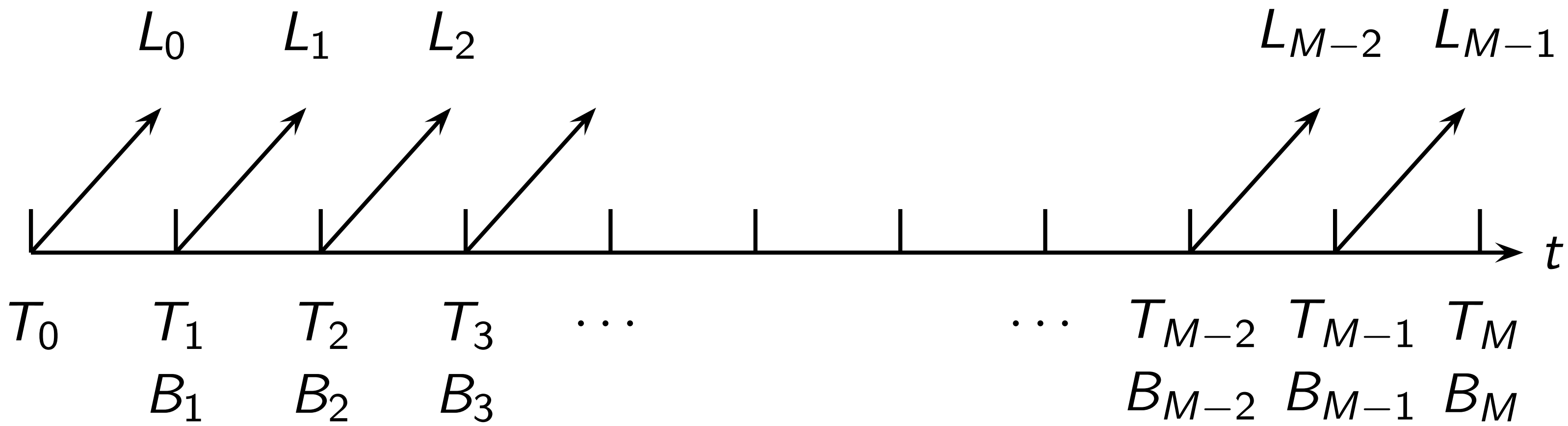
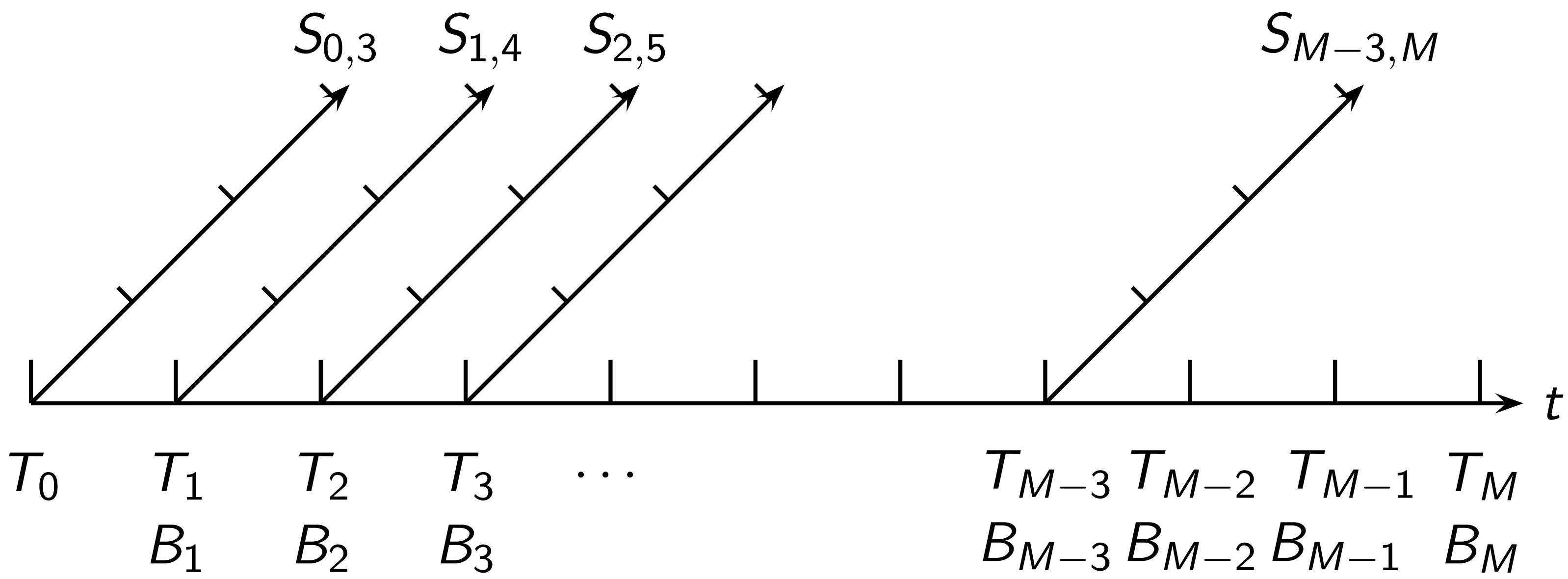
3.3 Retrieving Zero-Bond Prices



The tenor structure of Libor Market Model (LMM)

In LMM every zero-bond price $B_i(t)$ is uniquely determined by the simulated Libors $L_j(t), j < i$, LMM is then complete and admissible.



The tenor structure of Generalized Swap Market Model (GSMM) based on swap rates of 3 time periods.

In SGMM, zero-bond price $B_i(t)$ can not directly given by the simulated $S_{j,j+c}(t), j < i$, GSMM is then not admissible.

Second step to build up a complete GMM is to retrieve the zero-bond prices at times $(T_1, T_2, \cdots, T_M)$. Given $(M - c - j + 1)$ swap rates at time T_j, we have to bootstrap $(M - j)$ zero-bond prices at that time point.

Problem: The number of the swap rates is smaller than the number of zero-bond prices, GSMM is not a complete model.

We have generally two ways to overcome this issue.

1. Simple one-parameter bootstrapping.

(a): We suggest a functional form for $\tilde{B}_h(T_j)$ as follows

$$\tilde{B}_h(T_j) = e^{-\beta(T_h - T_j)} \tag{3.7}$$

with β as a constant. Plugging this zero-bond price form in $\tilde{S}_{M-c}(T_j)$ yields

$$\tilde{S}_{M-c}(T_j) = \frac{e^{-\beta(T_{M-c}-T_j)} - e^{-\beta(T_M-T_j)}}{\sum_{h=M-c+1}^{M} \tau_h e^{-\beta(T_h-T_j)}}, \tag{3.8}$$

We can solve the above equation for the parameter β quickly by using simple numerical routine, and are able to determine the zero-bond prices $(\tilde{B}_{M-c}(T_j), \tilde{B}_{M-c+1}(T_j), \cdots, \tilde{B}_M(T_j))$.

(b): Now it is easy to retrieve $\tilde{B}_{M-c-1}(T_j)$ by using the swap rate $\tilde{S}_{M-c-1}(T_j)$,

$$\tilde{S}_{M-c-1}(T_j) = \frac{\tilde{B}_{M-c-1}(T_j) - \tilde{B}_{M-1}(T_j)}{\sum_{h=M-c}^{M-1} \tau_h \tilde{B}_h(T_j)}.$$

By rearranging the above equation we obtain directly $\tilde{B}_{M-c-1}(T_j)$

$$\tilde{B}_{M-c-1}(T_j) = \tilde{S}_{M-c-1}(T_j) \sum_{h=M-c}^{M-1} \tau_h \tilde{B}_h(T_j) + \tilde{B}_{M-1}(T_j), \tag{3.9}$$

where all terms on the right hand of the above equation are known. Repeating this procedure results in a bootstrapped yield curve. Based on the retrieved yield curve we can construct any other swap rates as well as Libors.

2. Model-based bootstrapping.

We propose the zero-bond prices take the analytical form of some short rate models. For example, the functional form of $\tilde{B}_h(T_j)$ could the following one as in Vasicek model (1977),

$$\tilde{B}_h(T_j) = a(\delta_{hj}) e^{-b(\delta_{hj}) r(T_j)}, \qquad \delta_{hj} = T_h - T_j.$$

with

$$b(\delta_{hj}) = \frac{1}{\kappa}\left[1 - e^{-\kappa\delta_{hj}}\right],$$

$$a(\delta_{hj}) = \exp\left[\left(\theta - \frac{\sigma^2}{2\kappa^2}\right)[b(\delta_{hj}) - \delta_{hj}] - \frac{\sigma^2}{4\kappa}b^2(\delta_{hj})\right].$$

The dynamics of the short rate $r(t)$ in Vasicek model is given by

$$dr(t) = \kappa[\theta - r(t)]dt + \sigma dW(t).$$

This means we are transforming GSMM into Vasicek model in the terms of zero-bond prices. Note the parameters κ, θ, σ should hold for every zero-bond prices. We can estimate them with the initial yield curve at time T_0. As κ, θ, σ known, we only need to estimate the stochastic rate $r(T_j)$ at time T_j using swap rate $\tilde{S}_{M-c}(T_j)$. The estimation procedure is identical to the one for β above.

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Given the zero-bond prices at every time, we can construct any interest rate index, for example, Libors. GSMM is then completely established.

3.4 Simulation and Valuation

3.4.1 Simulation

Generally, the simulation of GSMM is identical to LMM. The Euler discretization of GSMM is

$$S_i(T_{h+1}) = S_i(T_h) + S_i(T_h)[\mu_i(T_h)\tau_{h+1} + \sigma_i(T_h)Z_i(T_h)\tau_{h+1}], \tag{3.10}$$

where $Z_i(T_h), h \leq i \leq M-c$, are some correlated standard normal distributed random variables drawn at time T_h. A more sophisticated simulation is based on the process of $\ln S_i(t)$ and is given by

$$S_i(T_{h+1}) = S_i(T_h)\exp[(\mu_i(T_h) - \frac{1}{2}\sigma_i^2(T_h))\tau_{h+1} + \sigma_i(T_h)Z_i(T_h)\tau_{h+1}]. \quad (3.11)$$

The piece-wise constant term volatility $\sigma_i(t)$ can be retrieved from the implied volatility $\bar{\sigma}_i$ of co-sliding swaptions. Neglecting the smile effect of $\sigma_i(t)$, we have the following relation between $\sigma_i(t)$ and $\bar{\sigma}_i$,

$$\bar{\sigma}_i^2 T_i = \int_0^{T_i} \sigma_i^2(t)dt = \sum_{h=1}^{i} \sigma_h^2(T_{h-1})\tau_h.$$

$\sigma(t)$ are applied to simulate the simple GSMM. A popular way to obtain a

smooth curve for $\sigma(t)$ is the following parameterization with four parameters a_0, a_1, a_2, a_3,

$$\sigma(T_i) = a_0 + (a_1 T_i + a_2)e^{a_3 T_i}, \tag{3.12}$$

which allows for humped, increasing and decreasing term structures.

Note, the above approach to retrieving term vola for co-sliding swaptions is only valid for GSMM and LMM, not for co-terminal and co-initial SMM. This shows only GSMM coincides with the mathematical structure of LMM.

3.4.2 Valuation

Let $G(T_j)$ denote the payoff function at time T_j of an interest rate derivative, it is a function of swap rates $S_1(T_1), S_2(T_2), \cdots, S_j(T_j), S_{j+1}(T_j), \cdots, S_{M-c}(T_j)$. The discounted payoff of $G(T_j)$ at time $T_p, p < j$, under risk-neutral

measure is

$$\frac{G(T_j)}{M(T_p,T_j)} = \frac{G(T_j)M(T_p)}{M(T_j)}.$$

The value of this derivative at time T_p under the swap measure $Q^k, k \geq j$, reads

$$G(T_p) = M(T_p)\mathbf{E}^Q\Big[\frac{G(T_j)}{M(T_j)}\Big] = N_k(T_p)\mathbf{E}^{Q^k}\Big[\frac{G(T_j)}{N_k(T_j)}\Big] \tag{3.13}$$

where we use $\frac{dQ}{dQ^k}$ again to allow us price G under the given swap measure. This pricing equation can be used to price path-dependent interest rate structures. Particularly we have the present value of $G(T_j)$ at time 0

$$G(0) = N_k(0)\mathbf{E}^{Q^k}\Big[\frac{G(T_j)}{N_k(T_j)}\Big], \qquad k \geq j. \tag{3.14}$$

- A standard swaption with payoff $G(T_n) = N_n(T_n)[S_n(T_n) - K]^+$ is valued as

$$SW_{Payer}(0) = N_k(0)\mathbf{E}^{Q^k}\left[\frac{N_n(T_n)[S_n(T_n) - K]^+}{N_k(T_n)}\right]$$

which is simplified for $n = k$ to

$$SW_{Payer}(0) = N_n(0)\mathbf{E}^{Q^n}[[S_n(T_n) - K]^+].$$

This is equivalent to Black76 formula.

Some facts on the pricing issue in GSMM,

(i) The correlation ρ_{ij} between W_i and W_j does not affect the price of swaptions.

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(ii) GSMM is especially suited to CMS products, for example, CMS memory, CMS range arruals. We can input the correlation information to GSMM to achieve more reasonable pricing for correlation-sensible products. In contrast, if we price these products in LMM, the correlations between CMS rates are not transparent.

(iii) Even for CMS spread products, GSMM has a clear advantage over LMM. For example of a CMS5 - CMS2 structure, LMM must be so calibrated to meet the volatilities of CMS5 and CMS2, and to arrive at a plausible correlation between them via Libor correlations. This is a big challenge. But a GSMM based on 2y swap rates needs only to meet the volatilities of CMS5. We have one problem less in GSMM than LMM.

(iv) The risk sensitivities regarding swap rates in GSMM are more stable, transparent and efficient than in LMM. For example, to obtain a reasonable

- Vega of CMS2 in LMM is a mission impossible (If possible, have to be expensive).

4 Smile Modeling for SMM

Generally all smile models extending LMM can be taken over to set up smile models within GSMM. To illustrate how to establish a smile GSMM analogy to the smile LMM, we list some possible extensions for SMM:

1. Local Volatility Model á la Andersen and Andreasen (1998). $S_i(t)$ follows a displaced constant elasticity of variance (CEV) process under the respective swap measure Q^i

$$dS_i(t) = [S_i^b(t) - a]\sigma_i(t)dW_i(t), \qquad a > 0, 0 \leq b \leq 1. \tag{4.1}$$

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that will becomes under spot swap measure Q^1

$$dS_i(t) = [S_i^b(t) - a][\mu_i(t) + \sigma_i(t)]dW_i^{Q^1}(t), \qquad a > 0, 0 \leq b \leq 1. \tag{4.2}$$

with

$$\mu_i(t) = \sum_{j=1}^{i} \frac{\tau \rho_{ij} \sigma_j \sigma_i [S_i^b(t) - a]}{1 + \tau S_i(t)}.$$

Since both displacement and CEV produce only skew effect, this smile modeling of GSMM does not allow symmetric smile pattern.

2. Stochastic Volatility Model á la Andersen and Brotherton-Raticiliff (2001). In this smile formulation, $S_i(t)$ has a mean-reverting squared root

process $V(t)$ as a part of variance under the respective swap measure Q^i,

$$\begin{aligned} dS_i(t) &= \phi(S_i(t))\sigma_i(t)\sqrt{V(t)}dW_i(t), \\ dV(t) &= \kappa(\theta - V(t))dt + \nu\sqrt{V(t)}dZ(t), \end{aligned} \tag{4.3}$$

where $W_i(t)$ and $Z(t)$ are uncorrelated. Due to zero correlation between swap rates and its stochastic volatility component we need additional mechanism to generate a skew effect.

3. Local/Stochastic Volatility Model á la Piterbarg (2003). This smile modeling is a mixture of local volatility and stochastic volatility. $S_i(t)$ follows the following process under the respective swap measure Q^i,

$$\begin{aligned} dS_i(t) &= [b_i S_i(t) + (1 - b_i)S_i(0)]\sigma_i(t)\sqrt{V(t)}dW_i(t), \\ dV(t) &= \kappa(\theta - V(t))dt + \nu\sqrt{V(t)}dZ(t), \end{aligned} \tag{4.4}$$

where $W_i(t)$ and $Z(t)$ are uncorrelated. This model can also be regarded as a special case of Andersen and Brotherton-Raticiliff model, and introduces displacement to generate volatility skew.

4. Stochastic Volatility Model á la Wu and Zhang (2002). This smile model takes the following form under the respective swap measure Q^i,

$$\begin{aligned} dS_i(t) &= S_i(t)\sigma_i(t)\sqrt{V(t)}dW_i(t), \\ dV(t) &= \kappa(\theta - V(t) - \xi_i(t))dt + \nu\sqrt{V(t)}dZ(t), \end{aligned} \tag{4.5}$$

where $\xi_i(t$ is a drift adjustment due to the change of risk-neutral measure to measure Q^i. $W(t)$ and $Z(t)$ are correlated,

$$\frac{< \sigma_i(t)dW_i(t) > dZ(t)}{|\sigma_i(t)|} = \rho_i dt.$$

All swap rates share a single volatility process.

5. Stochastic Volatility Model á la Zhu (2007). Zhu proposes a smile modeling for LMM with individual stochastic volatility process and individual volatility correlation. This model can be applied for $S_i(t)$ under respective swap measure Q^i,

$$\begin{aligned} dS_i(t) &= S_i(t)\sqrt{V_i(t)}dW_i(t), \\ dV_i(t) &= \kappa_i(\theta_i - V(t))dt + \nu_i\sqrt{V_i(t)}dZ(t), \end{aligned} \tag{4.6}$$

with $dW_i(t)dZ(t) = \rho_i$. The parameters κ_i, θ_i, ν_i and $V_i(0)$ are parameterized by a nesting function respectively, which allow for a parsimonious formulation of this smile model.

5 Conclusions

As discussed, we draw some conclusions as follows,

(i) GSMM models swap rates directly, and therefore achieves the best match between products and model, and especially suited for CMS products.

(ii) GSMM can be calibrated to the term structure of co-sliding swaption volatilities easily and quickly.

(iii) There is no translation of risk sensitivities with respect to swap rates within GSMM. In contrast, risk sensitives such as Vega for swap rates can not be derived directly, and must be translated in an inefficient, inaccurate and nontransparent manner in the most existing interest rate models.

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(iv) All smile modelings for LMM can be taken over for GSMM since GSMM and LMM share an almost identical mathematical structure.

(v) GSMM performs faster and more efficiently than LMM if we are pricing swap rate products.

(vi) GSMM avoids the inconsistency of the market conventions in cap and swaptions markets.

Therefore, GSMM should be a promising interest rate model for pricing and hedging most traded swap rate structures in financial market.

Thanks!!!

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